The Sixth
Clemson mini-Conference
on
Discrete Mathematics

Funded by the Office of Naval Research (ONR)

Conference Program and
Invited Talk View Graphs

October 3-4, 1991

Clemson University
Clemson, South Carolina

19941221 080

The Sixth
Clemson mini-Conference
on
Discrete Mathematics

Funded by the Office of Naval Research (ONR)

October 3-4, 1991

Clemson University
Clemson, South Carolina

Organizers: S.T. Hedetniemi   Department of Computer Science
             R.C. Laskar   Department of Mathematical Sciences
             R.D. Ringeisen   Department of Mathematical Sciences

Conference Program and
Invited Talk View Graphs

The Sixth Clemson mini-Conference

on

Discrete Mathematics

Clemson, South Carolina
October 3-4, 1991

Schedule of talks
(All talks given in Student Senate Chambers)

Thursday, October 3

1:30 - 2:00   *Sign-In/Registration*

2:00 - 2:05   *Welcoming Remarks* by R.D. Ringeisen, S.T. Hedetniemi, and Dr. Bobby Wixson, Dean of College of Sciences

2:05 - 2:50   *Marc J. Lipman, Office of Naval Research*
Mathematical Science Division, Arlington, VA

"Sphere-of-Influence Graphs"

An introduction to a class of geometrically defined objects, sphere-of-influence graphs (SIGs), used by the computer vision community to capture the low-level perceptual structure of a scene, that is, for pattern recognition. The mathematics of SIGs isn't yet mature.

2:55 - 3:35   *Roger Entringer, University of New Mexico*
Dept. of Mathematics, Albuquerque, NM

"Two Extremal Problems In Graph Theory"

Two specific instances of the following general problems are addressed:

(i) How many edges can a graph G of order n have if G must have a specific property?

(ii) If G is to have order n and a given number of edges, how are the edges arranged if a specific property must be optimized?

The instance of the second problem involves an attempt to shorten delivery time by the USPS.

3:35 - 4:00   *BREAK*

4:00 - 4:40

Edward R. Schelnerman, *Johns Hopkins University*
Department of Mathematical Sciences, Baltimore, MD

"Containment Orders and Planar Graphs"

We will explore interesting relationships between the worlds of partially ordered sets (especially *containment orders*) and planar graphs.

Given any graph $G = (V, E)$, let its *vertex-edge incidence order*, $P(G)$, be the partially ordered set whose ground set is $V \cup E$ together with relations $v < e$ exactly when $v \in V, e \in E$, and v is an end point of e. What *graph* properties of G can we deduce from *poset* properties of $P(G)$?

In this talk we will focus on order theoretic properties of $P(G)$ which turn out to be equivalent to G being planar. We will phrase these properties in terms of geometric containment orders, which we now define.

Given a family Σ of objects, we call a partially ordered set $P = (X, \leq)$ a Σ-order provided we can assign to each $x \in X$, an element $S_x \in \Sigma$, so that $x \leq y$ iff $S_x \subseteq S_y$. In particular, if Σ is the set of disks (circles with their interiors) in the plane, then Σ-orders are also known as *circle orders*.

We will discuss a number of results, all of which have the following flavor.

Theorem. A graph G is planar if and only if its vertex-edge incidence order, $P(G)$, is a circle order. $\square$

Some of the theorems to be presented include joint work with Graham Brightwell, Ann Trenk and Daniel Ullman.

4:45 - 5:25

Jean R.S. Blair, *University of Tennessee*
Dept. of Computer Science, Knoxville, TN

"On Finding Transmitter-Receiver Matchings"

The problem of finding a maximum transmitter-reciever matching (TRM) in communication networks is addressed. TRM remains NP-complete even for networks whose topologies correspond to chordal graphs. We address the problem for a subclass of chordal graphs, namely those graphs whose clique graphs are acyclic. Using several interesting properties of these graphs, we devise a linear time algorithm to solve the problem.

7:30

Social, Jordan Room

Friday, October 4

8:30 - 9:10

J. Chris Fisher, *University of Regina, Canada*
(Visiting Clemson University, Dept. of Math. Sci.)

"The Jamison Method in Galois Geometries"

In a fundamental paper Robert E. Jamison showed, among other things, that any subset of the points of AG(2,q) — the affine plane of order q — that intersects all lines contains at least 2q-1 points. Here I shall discuss my recent work with Aiden Bruen in which we show that Jamison's method of proof can be applied to several other basic problems in finite geometries of a varied nature. These problems include the celebrated flock theorem and also the characterization of the elements of GF(q) as a set of squares in $GF(q^2)$ with certain properties. This last result, due to A. Blokhuis, settled an important conjecture due to J.H. van Lint and the late J. MacWilliams.

9:15 - 9:55

Fred S. Roberts, *Rutgers University*
Dept. of Mathematics, Center of Operations Research (RUTCOR), and
Center for Discrete Mathematics and Theoretical Computer Science (DIMACS)
New Brunswick, NJ

"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

In the past five years, problems of the uniqueness of scales of measurement have been giving rise to a variety of interesting sequences of positive integers with fascinating combinatorial properties. Examples of such sequences are all non-decreasing sequences of positive integers $x_1, x_2, ..., x_n$ so that $x_1 = x_2 = 1$. Such a sequence is called *elementary* if all $k \leq n$, $x_k > 1$ implies that $x_k = x_i + x_j$ for some $i \neq j$. It is called *sub-Fibonacci* if $x_k \leq x_{k-1} + x_{k-2}$, $k = 3, 4, ...$ It is called *regular* if $x_j \leq \sum_{i=1}^{j-1} x_i$, $j = 3, 4, ...$ A regular sequence is called *Van Lier* if for all $j < k \leq n$, there is a subset A of $\{1, 2, ..., n\}$ with j not in A and $x_k - x_j = \sum_{i \in A} x_i$. We discuss these and other sequences and some of their combinatorial properties.

9:55 - 10:20   **BREAK**

10:20 - 11:00

Michael S. Jacobson, *University of Louisville*
Department of Mathematics, Louisville, KY

"Generating k-element Subsets of an n-element Set"

In this talk, a generalization of the idea of De Bruijn graphs will be used to establish sequences which generate all k-element subsets of an n-element set. In the case when n is odd, by using a result of Good, these sequences are shown to exist. When n is even, the technique shown will not generate an appropriate sequence. In fact the generalized De Bruijn graph is disconnected, and by a unique application of Polya's Theorem, the number of components of this graph is calculated.

11:05 - 11:45

E. Rodney Canfield, *University of Georgia*
Dept. of Computer Science, Athens, GA

"Matchings in the Partition Lattice"

Let $[n]$ be the set $\{1, 2, ..., n\}$. A *partition* of $[n]$ is a set of nonempty, pairwise disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. Partition π_1 is a *refinement* of partition π_2, denoted $\pi_1 \leq \pi_2$, provided each block of π_1 is contained in a block of π_2. Under this ordering the set of partitions P_n forms a lattice. The subcollection of partitions $P_{n,k} \subseteq P_n$ which have exactly k blocks has cardinality $S(n,k)$, the Stirling number of the second kind. The Stirling numbers are unimodal, raising the question of decomposing P_n into disjoint chains, $S(n, K_n)$ in number, $S(n, K_n)$ being $\max_k S(n, k)$. Our topic in this talk: for what k is it possible to find a *matching* of $P_{n,k}$ into $P_{n,k\pm1}$? That is, to find a one-to-one function $\varnothing$ from $P_{n,k}$ into $P_{n, k\pm1}$ with the property that π and $\varnothing(\pi)$ are comparable under the refinement relation "$\leq$".

LUNCH

1:15 - 1:55

Ronald C. Read, *University of Waterloo*
Dept. of Combinatorics and Optimization, Ontario, Canada

"Algorithms for Small Graphs"

The compilation of an "Atlas" of graph theory - a project that I am working on with R.J. Wilson - has called for the computation of many invariants (girth, connectivity, etc.) and properties (planarity, hamiltonicity, etc.) of large number of graphs; but the graphs themselves are quite small. Thus the usual concern about complexity of the algorithms is largely irrelevant, and the methods that will be used are often quite different from those that would be used for large graphs.

My talk describes some of this work. We shall see what graph theory algorithms look like through the wrong end of the telescope!

2:00 - 2:40

Nathaniel Dean, *Bellcore*
Morristown, NJ

"Characterization of Generalized Bicritical Graphs"

A recent theorem of Thomas and Yu states that every 4-connected projective planar graph is hamiltonian and, as a corollary, has a 2-factor. We extend this latter result by showing that the deletion of any vertex or two vertices of such a graph leaves a graph with a 2-factor. This result is in fact only an application of results we prove concerning f factors in graphs with removed elements and generalizes several notions in matching theory including bicritical graphs, i.e., where the deletion of any pair of vertices yields a graph with a perfect matching.

2:40 - 3:00   **BREAK**

3:00 - 3:40

Joseph Straight, SUNY at Fredonia
Dept. of Mathematics and Computer Science, Fredonia, NY

"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

Given a simple graph $G = (V,E)$, a subset S of V is called a *neighborhood set* provided G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. The minimum and maximum cardinalities among all minimal neighborhood sets of G are denoted by $n(G)$ and $N(G)$, respectively; $n(G)$ is call the *neighborhood number* of G. It is known, for instance, that $\gamma(G) \leq n(G) \leq \alpha(G)$, where $\gamma(G)$ and $\alpha(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $n(p)$ among all connected graphs of order p. Our work so far has lead us to conjecture that

$$n(p) \leq [9p/13]$$

a result that holds for $2 \leq p \leq 15$. I will report on this work and, as time permits, a number of other extremal problems, including some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order p having girth at least 5.

3:45 - 4:25

Andrzej Rucinski, Emory University
Dept. of Mathematics and Computer Science, Atlanta, GA

"Random Graph Processes with Degree Restrictions"

Suppose that a process begins with n isolated vertices, to which edges are added randomly one by one so that the maximum degree of the induced graph is always bounded above by d. We prove that if n approaches infinity with d fixed, then with probability tending to 1, the final result of this process is a graph with [nd/2] edges. For d = 2, the number of 1-cycles in this graph is shown to be asymptotically Poisson (1 > 2).

"Sphere-of-Influence Graphs"

Marc J. Lipman, Office of Naval Research
Mathematical Science Division, Arlington, VA

On Abstract
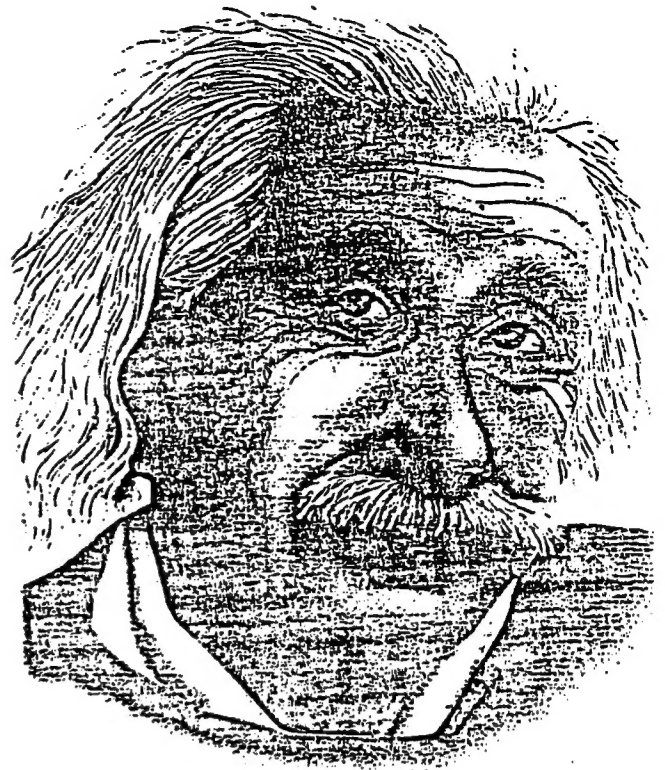Sphere - of - Influence
Graphs

Frank Harary

Michael S. Jacobson
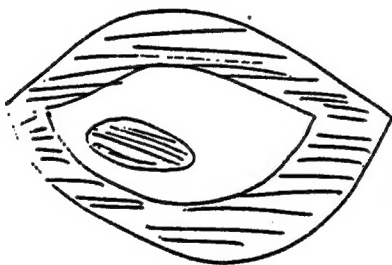
Marc J. Lipman

F. R. McMorris

1

"Perfection of means and confusion of goals
seem to characterize our age."

2

3

4

Let S be a finite set of at least two points in the euclidean plane. For each $x \in S$, let r_x be the smallest distance from x to any other point in S.

Let B_x be the open ball of radius r_x centered at x.

Let A_x be the closed ball of radius r_x centered at x.

The Sphere-of-Influence Graph of S, $G(S)$, has vertex set S, and for $x, y \in S$, x and y are adjacent in $G(S) \Longleftrightarrow$ $B_x \cap B_y \neq \emptyset$.

The Closed-Sphere-of-Influence Graph of S, $G^*(S)$, $\cdots$
$$\Longleftrightarrow \quad A_x \cap A_y \neq \emptyset.$$
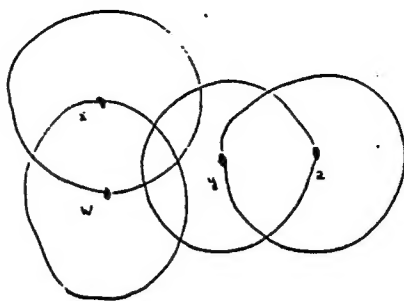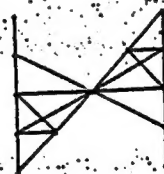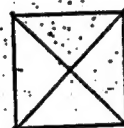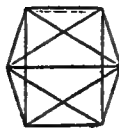

So, this is a SIG.

" " " " CSIG.

5

6

7

8
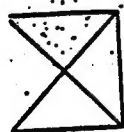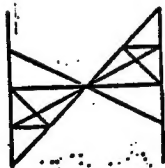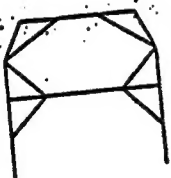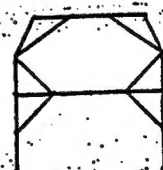
Thm: The graph $G(S)$ and $G^*(S)$ for a set S with $|S|=n$ can be computed in time $O(n \log n)$.

So what's the problem?

We don't know much about the graphs $G(S)$ and $G^*(S)$. Do they do what they appear to do?

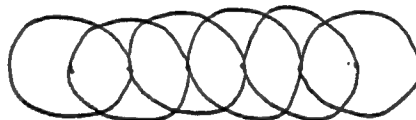Even more basic : Which graphs are SIGs?

Which graphs are CSIGs?

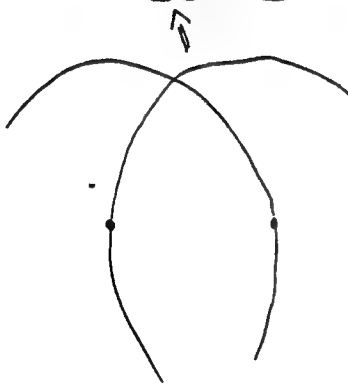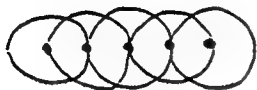Easy (important) result : The union of SIGs (CSIGs is a SIG (CSIG).
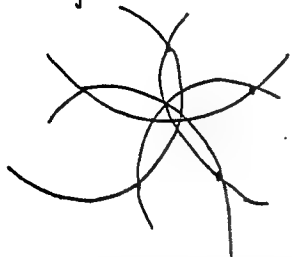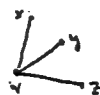

Thm: Paths are SIGs.

Thm: Cycles are SIGs.

Some Complete Graphs are SIGs.

Are there graphs which are not SIGs?

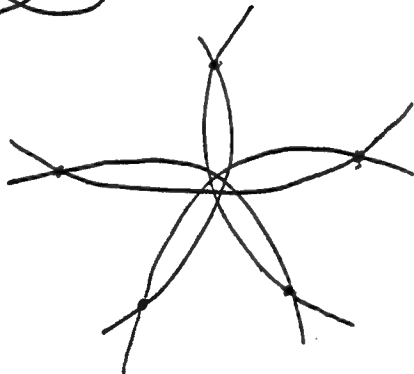$K_{1,3}$ is <u>not</u> a SIG.

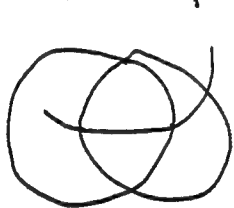**Thm [Erdős-Bateman]:** If $|S|=n$, then $G(S)$ has at most $17n$ edges.

So SIGs (and CSIGs) aren't to dense.

$$G(S) \neq k_6$$

$$G(CS) = k_6$$

Thm: The path on 3 vertices is not a CSIG.

[If $|S|=3$, then $G^*(S) \cong K_3$.]

Thm: Even paths are CSIGs. Odd ones aren't.

← NOT P_n (P_n^2)

Thm: The $\triangle$ and even cycles are CSIGs. The others aren't.
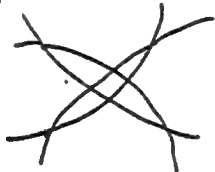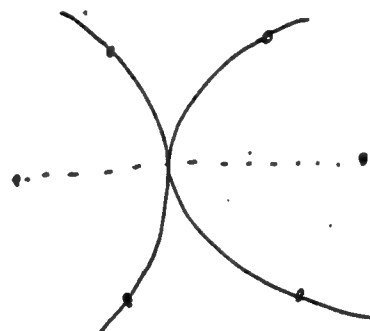
⭐ The class of SIGs (CSIGs) is **NOT** closed under taking subgraphs (or even taking induced subgraphs) !

→ There is no forbidden subgraph characterization of SIGs (CSIGs).

Thm: Every tree is the induced subgraph of a SIG.

Let us say that "x defines y" if x is a nearest neighbor of y (that is: $r_y = d(x,y)$)

* of course xy is then an edge in the SIG.

Theorem: If G is a CSIG without a $\triangle$, then G has a 1-factor of "defining edges."

(a 1-factor is a set of edges meeting every vertex EXACTLY once)

proof: If x defines y and z, then $\triangle$.

Otherwise: spose x_2 defines x_1,
$\qquad\qquad\qquad x_3$ defines x_2,
$\qquad\qquad\qquad\vdots$
$\qquad\qquad\qquad x_1$ defines x_k.

Then: $\qquad r_{x_2} \le d(x_2, x_1) \cdot r_{x_1}$
$\qquad\qquad\quad r_{x_3} \le d(x_3, x_2) \cdot r_{x_2}$
$\qquad\qquad\qquad \vdots$
$\qquad\qquad\quad r_{x_1} \le d(x_1, x_k) \cdot r_{x_k}$

$\Longrightarrow r_{x_1} = r_{x_2} = \cdots = r_{x_k} = d(x_m, x_i)$.

21

If $k > 2$, then x_2 defines x_1 and x_3 !

Therefore, $k = 2$ and x_1 and x_2 define each other
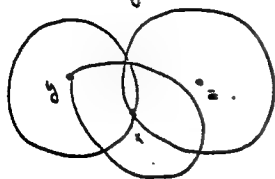(and no other points).

Therefore, the set of all "defining edges"
pairs up the points as above — and that
is the 1-factor.

* Note: The Theorem merely states that there
is a 1-factor. The proof shows that the full
set is the 1-factor.

COR: If G is a CSIG without a $\triangle$,
then G has an even number of points.

22

OSIGs are harder: P_3, for instance.

x defines both y and z !

23

However, if G is a tree, we get
a complete characterization:

Theorem: Suppose G is a tree. Then
$\quad$ i) G is a CSIG $\Longleftrightarrow$
$\qquad\qquad G$ has a 1-factor.

$\quad$ z) G is a OSIG $\Longleftrightarrow$
$\qquad\qquad G$ has a $\{P_2, P_3\}$-factor.

24

Ex:

25
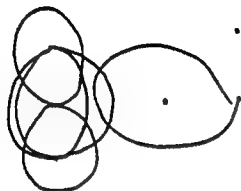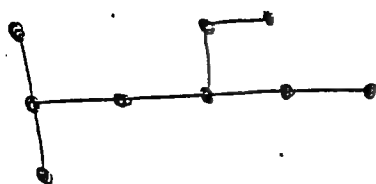
This can't work in general, since
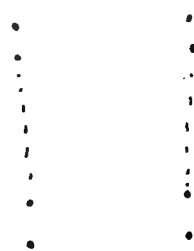
every possible edge
between $K_{n,n}$

has too many edges to be a SIG

but has a 1-factor and is $\triangle$-free.

26

$\bigcirc$uestion:

Since we are interested in computer vision, we really care about SIGs where the points have to show up in pixels.

Which SIGs show up here?

27

Theorem: If G is a OSIG, then G has a representation in which every point has <u>integer</u> coordinates.

Ex: $(0,0)$ • $(a,1)$ • $(20,0)$ •

C_5

/

$(0,-2a)$ • $(20,-2a)$ •

28

Idea: Spose G has a representation with no circles tangent:
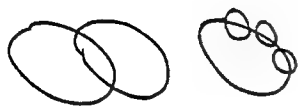
Then $\exists\ \epsilon > 0$ so that if any circle is moved $< \epsilon$ no intersections are changed.

Move the circles one at a time to _rational_ points.

Then "puff up" to integer points.

Oops: Sometimes you have to have tangencies: P_3, P_{odd} for OSIGs.

Fix: Spose only those, that is, no "accidental" tangencies:

$$P_3 \cup P_3$$

Then: move in "clumps"

If C_x and C_y are tangent at z, (so z defines them!) Then move all 3 circles together until z has rational coordinates.

Then rotate and move

along the xzy-line to fix x and y.

This works $\underset{\sim}{\overset{I F}{=}}$ such a representation exits.

---

does!

in.
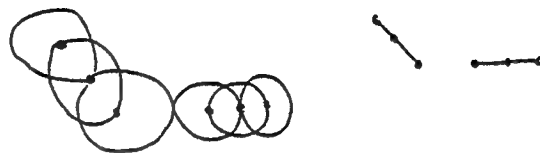
So:

1. We know a little about SIGs and CSIGs.

2. They _seem_ to be useful for object separation in some contexts.

3. They may be useful for object identification in similar contexts.

4. The definitions generalize to three dimensions and different geometries.

5. Our ignorance exceeds our knowledge.

33

"Two Extremal Problems in Graph Theory"

Roger Entringer, University of New Mexico
Department of Mathematics, Albuquerque, NM

C.A Barefoot
New Mexico Institute of Mining & Technology
Socorro, New Mexico 87801, USA

A.J. Depew*, L.H. Clark**, R.C. Entringer, A.A. Kooshesh***
and L.A. Székely****
University of New Mexico, Albuquerque, NM 87131, USA

*   Probably in the Cayman Islands by now.

**   Present   Department of Mathematics
       address: Southern Illinois University
              Carbondale, IL 62901

*** Should complete PhD in Computer Science this year -
     needs a job.

*** Permanent   Department of Computer Science
*   address: Eötvös Loránd University
                H-1088 Budapest
                HUNGARY

1

Q. How many edges can a graph of order n have if it doesn't contain a hamilton cycle?

A. (Ore 1961) $\binom{n-1}{2}+1$.

Q. What graphs have this many edges but don't contain a hamilton cycle?

A. K_{n-1} with a pendant vertex, $n \neq 5$.

Let $P(n)$ be a property enjoyed by K_n.

Generic Extremal Problem: *Determine the maximum number of edges, $ex(n;P(n))$, a graph of order n can have if it doesn't satisfy property $P(n)$.*

The graphs of order n that have $ex(n;P(n))$ edges but do not satisfy property $P(n)$ are called the *extremal graphs* for $P(n)$.

2

Q. How many edges can a graph of order n have if it doesn't contain any cycle?

A. $n-1$.

Q. What graphs have this many edges but don't contain a cycle?

A. Trees.

Q. How many edges can a graph of order n have if it doesn't contain a subdivision of K_4?

A. $ex(n;K_4S) = 2n-3$.

Q. How many edges can a graph of order n have if it doesn't contain a subdivision of K_5?

Conjecture (Dirac 1964) $ex(n;K_5S) = 3n-6$.

3

Fix the graph F. We define the *subdivision threshold* of F to be the maximum number of edges, $ex(n;FS)$, a graph of order n can have without containing a subdivision of F as a subgraph.

We denote by $EX(n;FS)$ the family of those graphs of order n that have $ex(n;FS)$ edges and do not contain a subdivision of F

A result of Mader shows that for any graph F there is a constant, c_f, such that $ex(n;FS) \leq c_f n$.

Theorem (Mader 1967). *If a graph has order n and size $2^{\binom{k}{2}-1} kn$ then it contains a subdivision of K_{1+k}.*

4

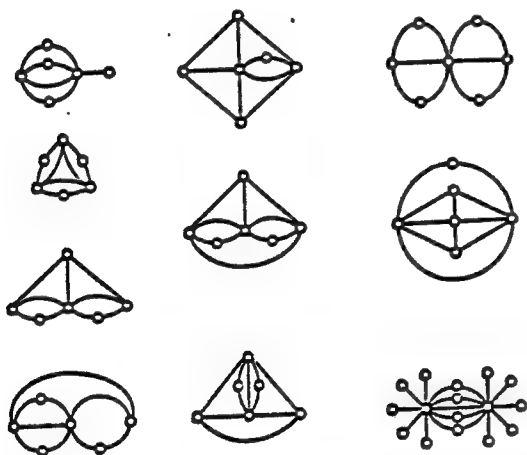**Theorem.** (Erdős and Pósa 1965) $ex(n;FS) = 3n - 6$. G is in $EX(n;FS)$ iff $G = K_3 + \overline{K}_{n-3}$.

Theorem. (BCDES) $ex(n;FS) = \begin{cases} 2n-2, & n \equiv 1 \mod 3 \\ 2n-3, & n \not\equiv 1 \mod 3 \end{cases}$

G is in $EX(n;FSR)$ iff every block of G, with at most one exception, B, is isomorphic to K_4 and $B = K_2 + \overline{K}$, or $B = K_{3,3}$ or $B = K_2 \times K_3$ or B is the nearly 3-regular graph of order 5.

Certain subgraphs of

+ pendant vertices.

Candidates for graphs G satisfying $ex(n;FS) < 3n - 6$.

Problem. *Find all graphs with subdivision threshold less than* $3n - 6$.

Properties of F when $ex(n;FS) < 3n - 6$:

(i) $\Delta(F) \leq 6$.

(ii) F has at most one vertex with degree ≥ 6.

(iii) F has at most two vertices with degrees ≥ 5.

(iv) F is planar.

(v)   If F is connected, then it has order ≤ 7.

(vi) If F is 2-connected, then it has order ≤ 6.

(vii) F is a subgraph of $K_3 + \overline{K}_{n-3}$.
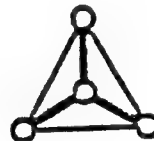
Theorem. (Thomassen 1974) $ex(n;FSR) = 2n - 3$. G is in $EX(n;FSR)$ iff G is a $(*,2)$-cockade where each member $*$ of the cockade is K_3 or $K_{3,3}$.
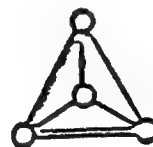
Theorem. (Krusenstjerna-Hafstrøm and Toft 1980) $ex(n;FSR) = 2n - 3$. G is in $EX(n;FSR)$ iff G is a $(3,2)$-cockade.

Theorem. (Erdős and Pósa 1965) $ex(n;FS) = 3n - 6$. G is in $EX(n;FS)$ iff $G = K_3 + \overline{K}_{n-3}$.

Theorem. (BCDEKS) $ex(n;FSR) = 3n - 6$. G is in $EX(n;FSR)$ iff $G = K_3 + \overline{K}_{n-3}$.

9

---

w/Kevin Burns University of New Mexico
Albuquerque, NM 87131, USA

33447

8 3 4 4 7   Squirrel, Idaho

3 3 4 4 7   Del Ray Beach, Florida

10

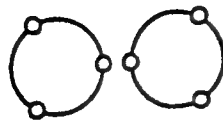The *load*, $L(v)$, of a vertex v of a tree T is the number of paths in T containing vertex v as an internal vertex. This has also been called the *cutting number* by Harary and Ostrand.

Suppose T has order n and that the branches of T at v have n_i edges, $1 \leq i \leq k$, (so that $\sum_{i=1}^{k} n_i = n - 1$) then

$$L(v) = \sum_{i \neq j} n_i n_j = \frac{1}{2}\left[(n-1)^2 - \sum_{i=1}^{k} n_i^2 \right].$$

The *load*, $L(T)$, of a tree T is the sum of the loads of the vertices.

11

The *transmission*, $\sigma(v)$, of a vertex v in a connected graph G is the sum of the distances from v to each of the remaining vertices of G.

The *transmission of a graph* G is defined by

$$\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} \sigma(v)$$

Observation.

$$L(T) = \sigma(T) - \binom{n}{2}$$

(since a path joining u and v contributes 1 to the load of each of $d(u,v) - 1$ vertices.)

The *transmission center* of a graph is the set of vertices with minimum transmission and consists of one vertex or two adjacent vertices. The transmission center is the centroid (Zelinka).

12

Theorem. *Of all trees of order n with exactly k end vertices,* $S(n,k)$ *has minimum transmission.*

$S(n,k)$:
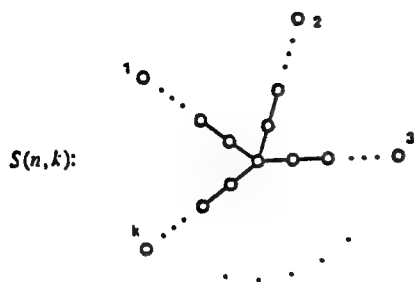
13

Theorem. *Of all trees of order n with exactly k end vertices,* $D(n,k)$ *has maximum transmission.*

$D(n,k)$:
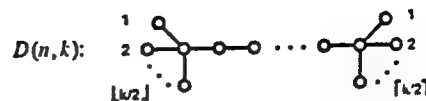
14
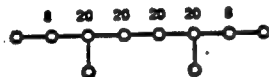
Question. *What fraction of the vertices of a tree of order n can have maximum load? In particular, does this fraction tend to 0?*
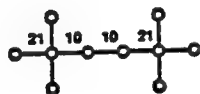
Question. *What fraction of the vertices of a tree of order n can have a relatively maximum load?*
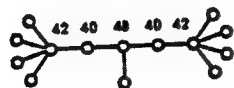
15

Problem. *Given a connected graph find a spanning tree with minimum transmission. For example, find a spanning tree of* Q_n *with minimum transmission.*

Theorem. *Let G be a k-partite graph with smallest part H. The spanning tree of G with minimum transmission contains two adjacent vertices, u in H and v, where u is adjacent to all vertices of G not in H and v is adjacent to all vertices of G in H.*

16

COLORADO

Denver

339 Km

Alamosa

45 Km

670 Km

Antonito

136 Km   Raton

152 Km

Las Vegas

Albuquerque   195 Km

NEW MEXICO

17
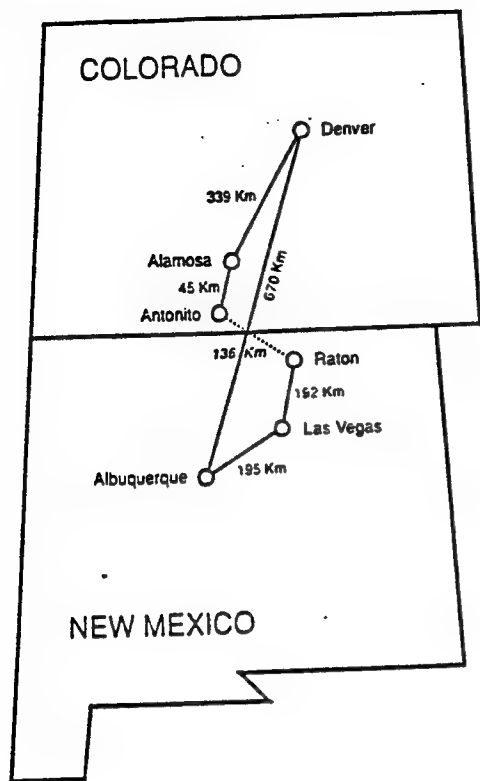
"Containment Orders and Planar Graphs"

Edward R. Scheinerman, Johns Hopkins University
Department of Mathematical Sciences, Baltimore, MD

Containment Planar Graphs Orders

Edward Scheinerman
Johns Hopkins University

1

Overview

⇶→ Circle Orders, and their relatives

⇶→ Building Posets from Graphs

⇶→ Graph Planarity ⇔ Poset Propertie
  "Triangle" Orders
  Circle Orders
  Point-Halfspace Orders
  Non-Planar Comments

⇶→ Planar Maps & Circle Orders

⇶→ The Circle Order Problem

2

Definitions...

Let P be a finite poset.
We call P a

provided we can assign to each $x \in P$
a
so that $x \le y$ iff

P

y z

x

3

---
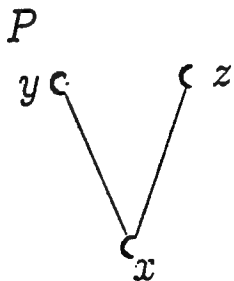
circle order

circle C_x

$$C_x \subseteq C_y$$

C_z

C_y C_x
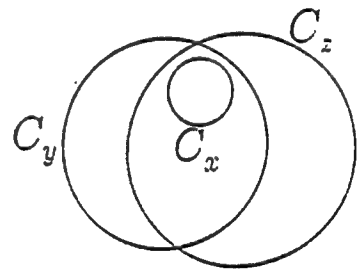
(also: S_i here *Orders* via balls in R^3)

4

parabola order

parabola P_x (upwards, filled)

$$P_x \subseteq P_y$$
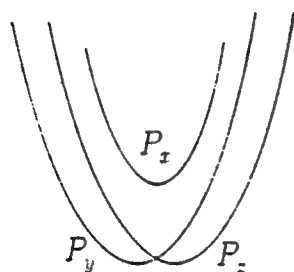
5

---

space-time* order

"event" P_x in space-time*

$$P_x \text{ precedes } P_y$$
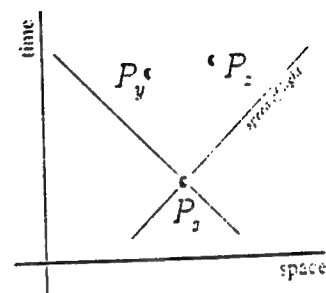
* two space coordinates, one time coordinate

6

RS2PD* order

real, symmetric, 2-by-2 matrix M_x

$$M_y - M_x \text{ is positive}$$

(semi) definite

$$M_x \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_y \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$M_z \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

* real, symmetric, two-by-two positive definite matrix order
(also, H2PD = complex Hermitian, two-by-two positive definite matrix order

7

What's the connection?

Theorem. The following statements about a finite poset P are equivalent:

- P is a circle order
- P is a space*-time
- P is a parabola order
- P is a RS2PD order

* two space coordinates

Theorem. The following statements about a finite poset P are equivalent:

- P is a sphere order
- P is a space*-time
- P is a H2PD order

* three space coordinates

But! Both statements are false for infinite posets.
[Brightwell & Scheinerman]

8

Why it works
(Q/L Orders)

Q/L order on R^{n+1}

$$\underbrace{x \leq y}_{\in R^{n+1}} \quad \Leftrightarrow \quad \begin{cases} Q(y-x) \geq 0 \quad \text{← a quadratic form} \\ L(y-x) \geq 0 \quad \text{← a linear form} \end{cases}$$

Theorem. The only possible Q/L orders on R^{n+1} are:
- anti-chain
- disjoin union of chains
- space*-time order (balls in R^n)

* n space coordinates   [Scheinerman]

9

Posets from Graphs

G   P(G)

10

What can you deduce about *graph* properties of G from *order* properties of P(G)?

Planarity

Graph Theory World

⇕

?

Partially Ordered Sets   11
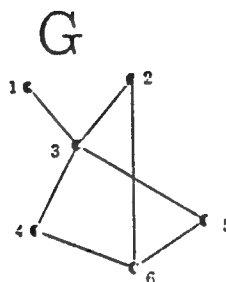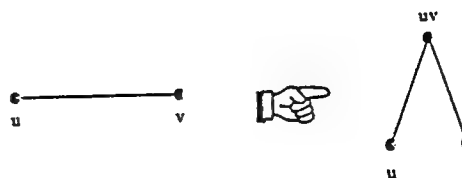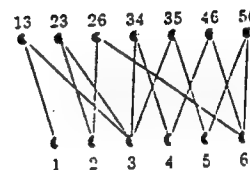
Theorem: A graph G is planar iff P(G) is a triangle* order. [Schnyder]

*equilateral triangles with bottom side parallel to the x-axis
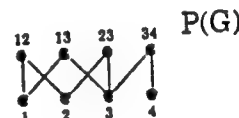
G   P(G)
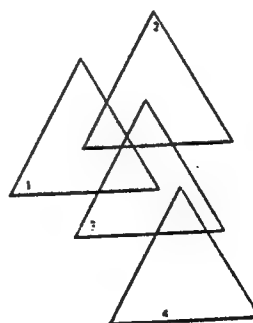
12

Theorem: A graph G is
planar iff P(G) is a circle
order. [Scheinerman]
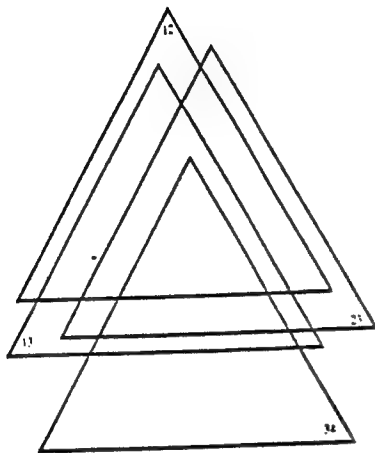
(Actual Statement: G is planar iff dim P(G) ≤ 3.) 13

Key Ideas in the Proof

#1 O.K. to work with the dual.

Ƅ(G)

G
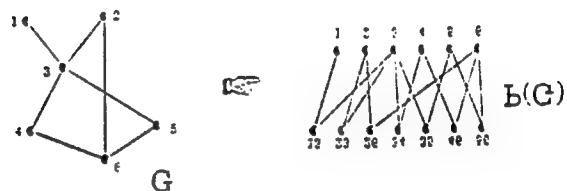
14

Key Ideas in the Proof
(continued)

#2 Thurston's Theorem.

Every planar graph has a representation by
disks in the plane...

D(v)

such that...

v   w
(adjacent)

D(v)   D(w)

externally tangent

v- - - - - - - w
(not adjacent)

D(v)   D(w)

disjoint 15

1
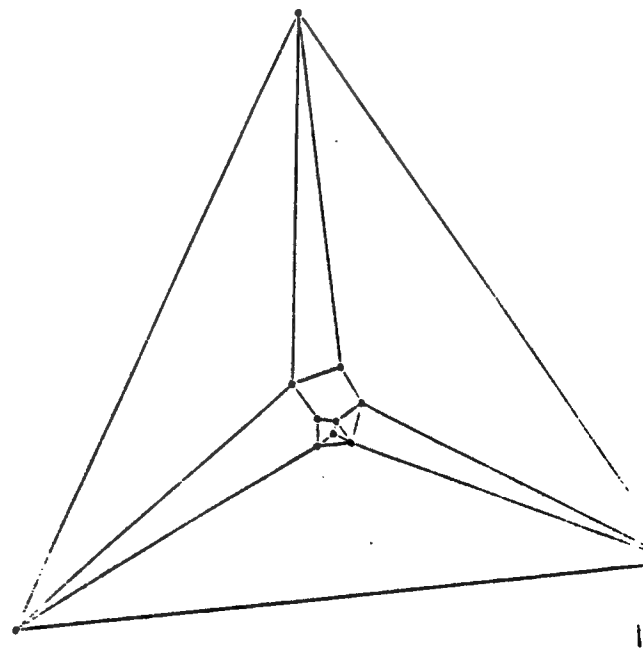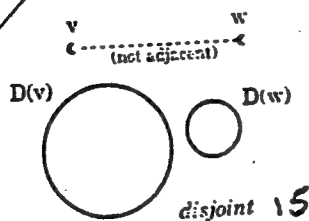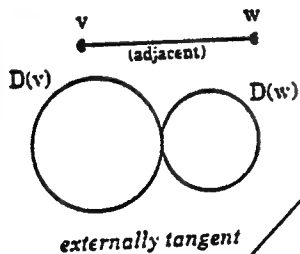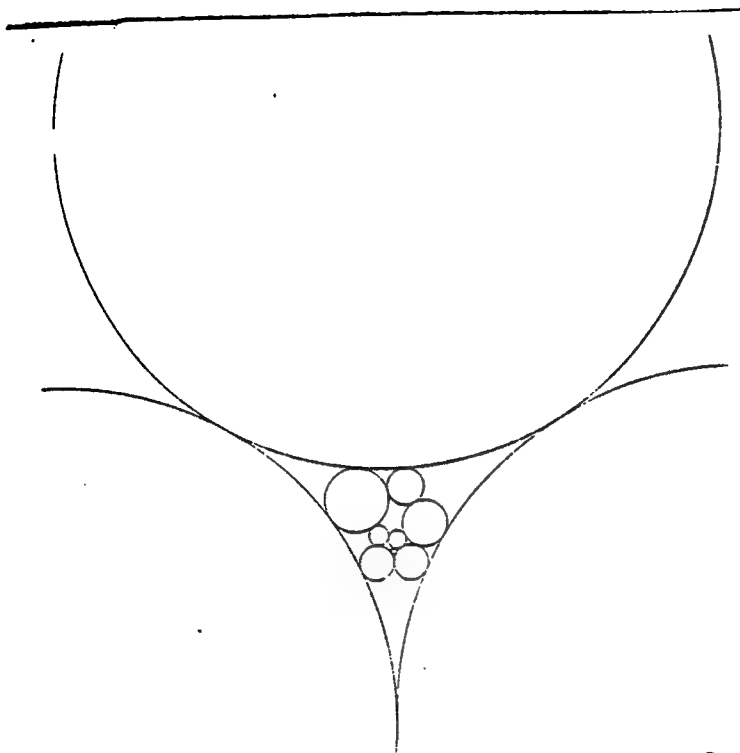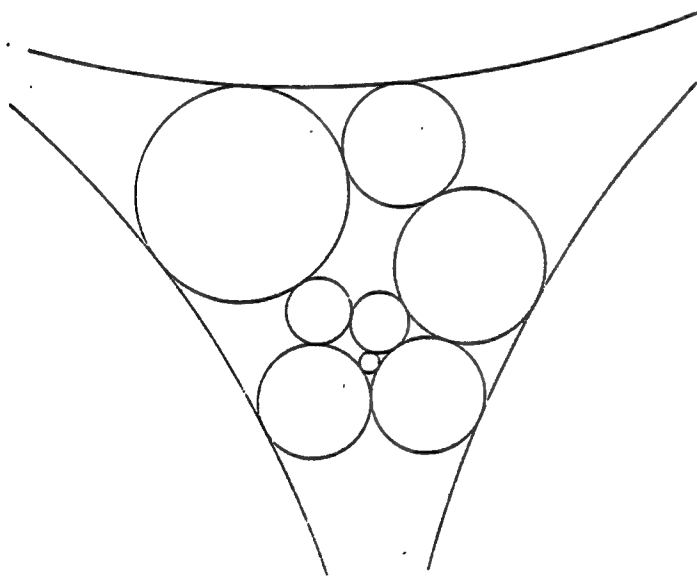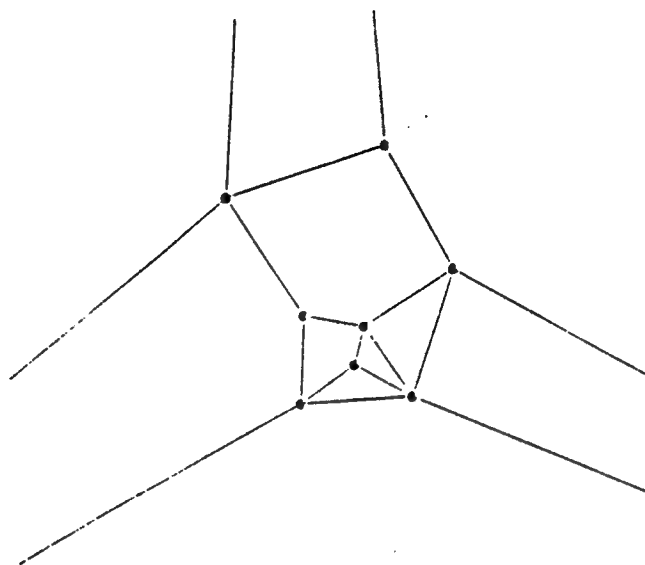
G planar $\Rightarrow$ P(G) is a circle orde

- Form Thurston circles for G. These will be the circles for V(G).

- Points of tangency will be the circles (of radius 0) for E(G).

- Notice: Every edge circle is contained in exactly its endpoints' circles.

This circle corresponds to a vertex...

...and this "circle" corresponds to an edge.
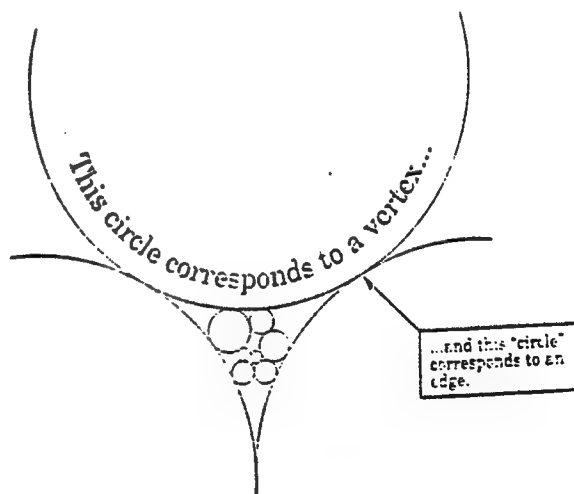
G planar ⇐ P(G) is a circle order

Draw the dual of P(G) as a circle order...
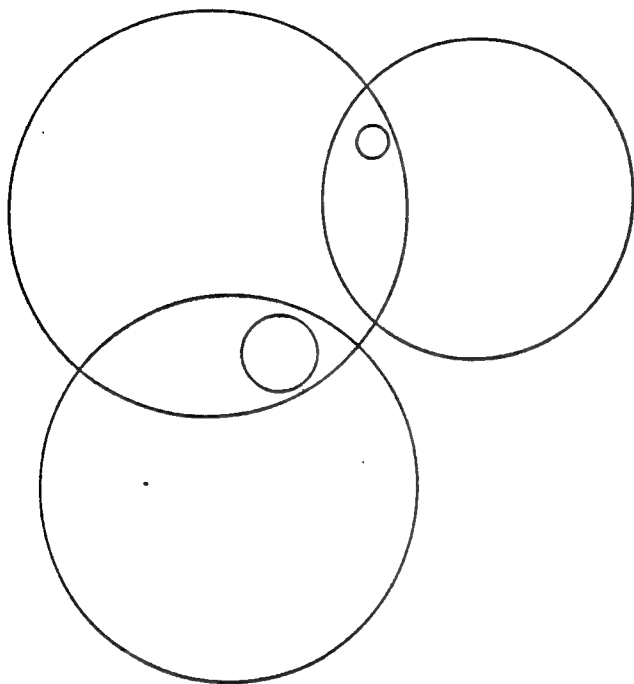
21
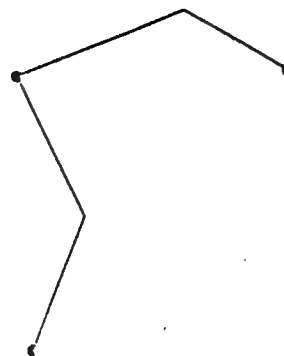
...and this will give
an embedding of G
in the plane!

22

Why 2-step paths might be needed

23
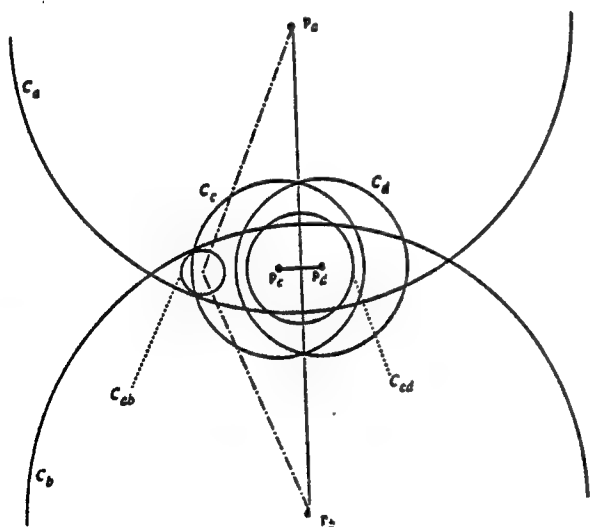
Point-Halfspace Orders

A "bipartite" poset is called a
point-halfspace order if...

P=P(G)

G
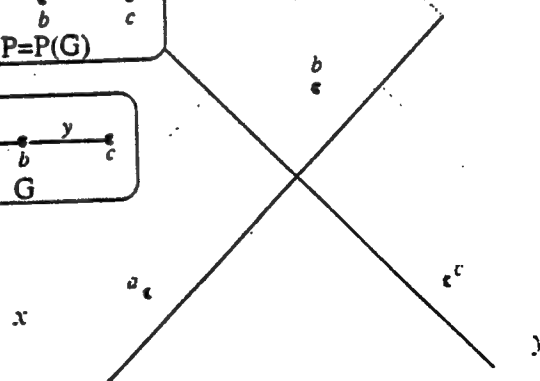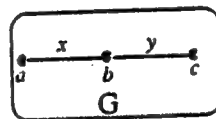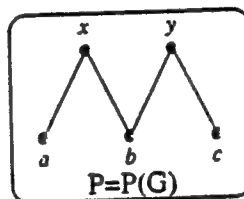
Theorem. Let G be a graph. P(G) is a point-halfspace
order in R^3 if and only if G is planar or K$_5$.

[Scheinerman, Trenk & Ullman] 24

For any graph G...

P(G) is a circle order

G is planar ⟺ P(G) is a triangle* order

P(G) is a point-halfspace order (and G ≠ K₄)
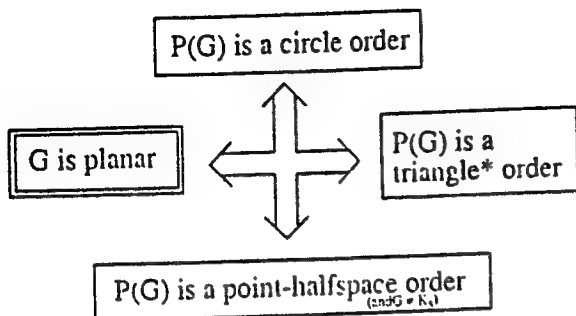
*Equilateral triangles with bottom parallel to x-axis. (Equivalent to dim P ≤ 3.)

25

What about non-planar graphs?

Theorem. If G is *any* graph, then P(G) is a sphere order. [Scheinerman]

Corollary. Let G be a graph. The least d such that G embeds in R^d equals the least d such that P(G) is representable by balls in R^d. [Scheinerman]

Theorem. If G is *any* graph, then P(G) is a point-halfspace order in R^4. [Scheinerman, Trenk & Ullman]

☞ Note: P(G) can have arbitrarily high poset dimension.

26

Planar Maps
(bounded faces only)
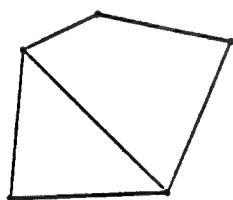
faces

edges

vertices

P(M)
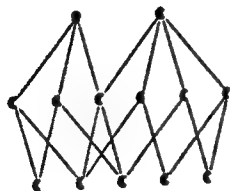
F(M)

E(M)

V(M)

Theorem. For any planar map, dim P(M) ≤ 3. (Brightwell & Trotter)

...but is P(M) a circle order?

27

Double Thurston Theorem

Theorem. (Brightwell & Scheinerman, Pulleyblank & Rote) If G is a 3-connected planar graph and G* is its dual, then we can make "Thurston Circles" for G and G* simultaneously so that...

circle for a vertex of G

circle for a vertex of G* (face of G)

And they cross at 90°

28

29
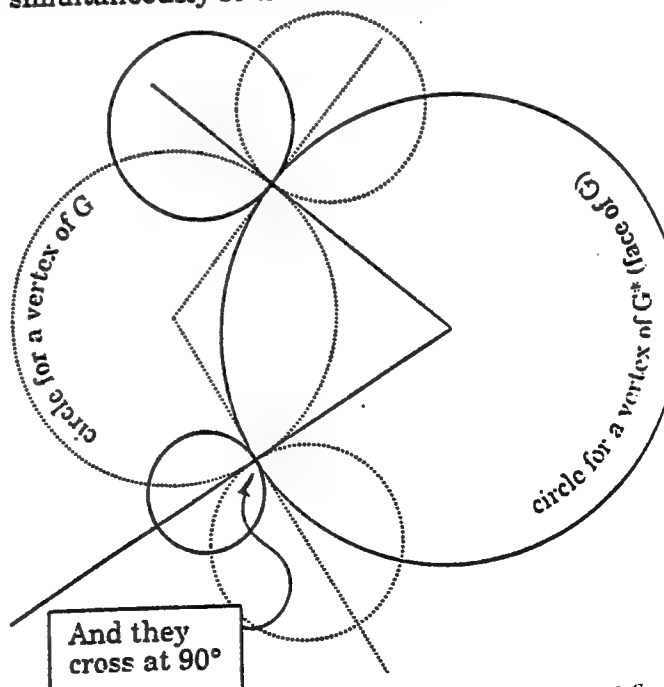
Adjusting Time & Circles of Negative Radius

Positive Radius

Zero Radius

Negative Radius

30
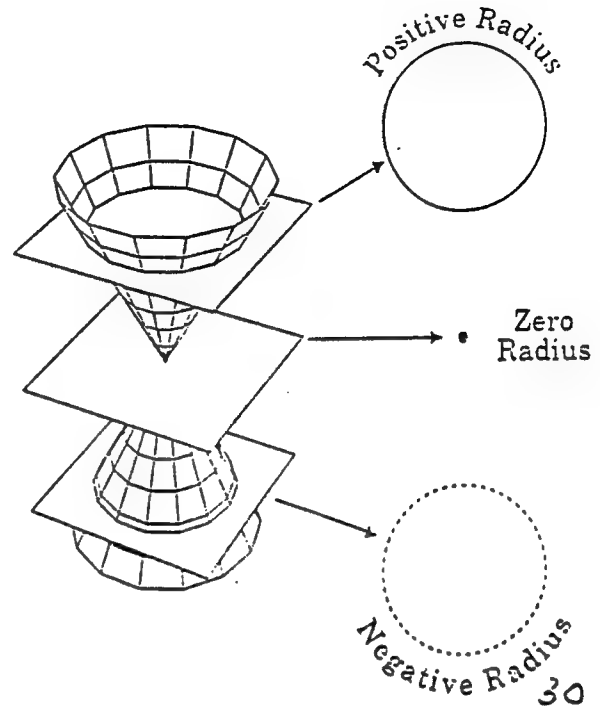
Containment of Positive and Negative Circles

$$A \supset B$$

A B

A B

A B

$+$ $-$
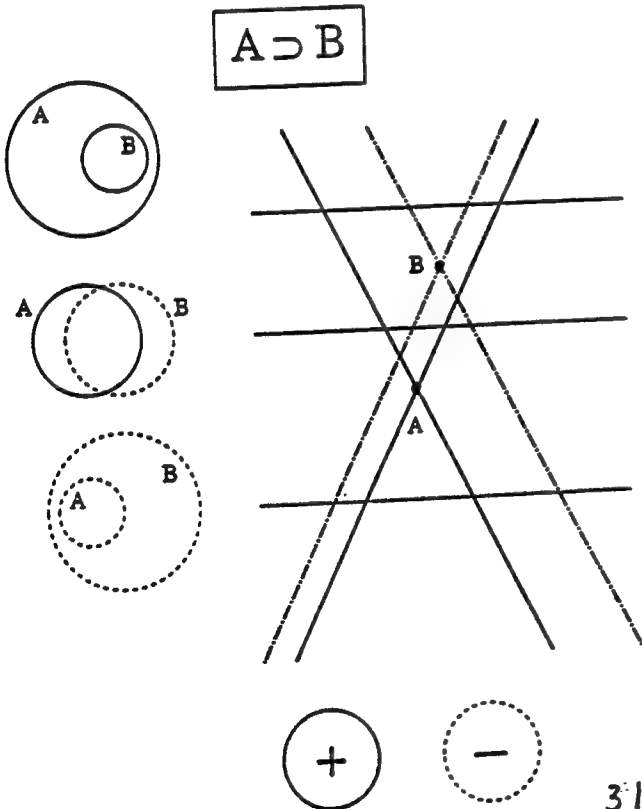
31

Main Theorem

Theorem. If M is a 3-connected planar map then P(M) is a circle order. (Brightwell & Scheinerman)

Proof...

Face Circle, Positive Radius

Edge Circle, Zero Radius

Vertex Circle, Negative Radius

Remember! When...

$-$ $+$

A   B

...A is contained in B

3̲

Why P(M) is a circle order

Edge Circle, Zero Radius

Vertex Circle, Negative Radius

Face Circle, Positive Radius
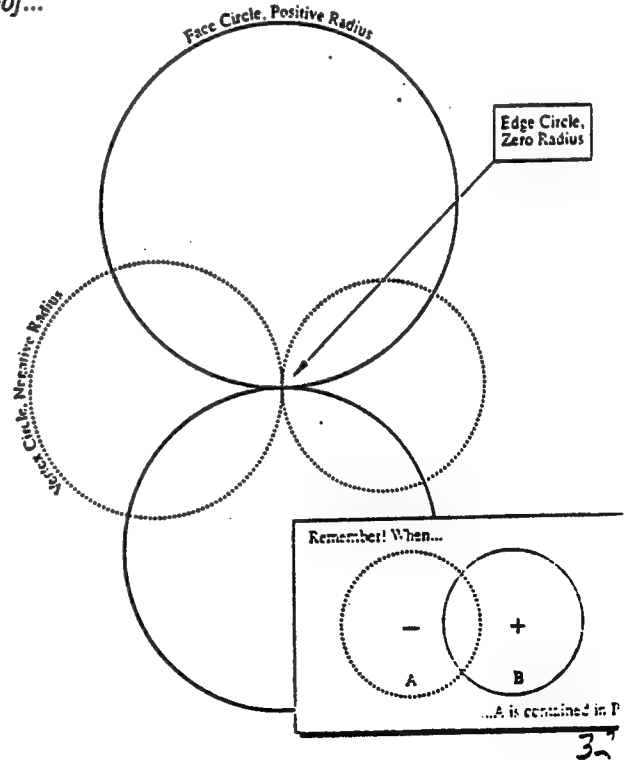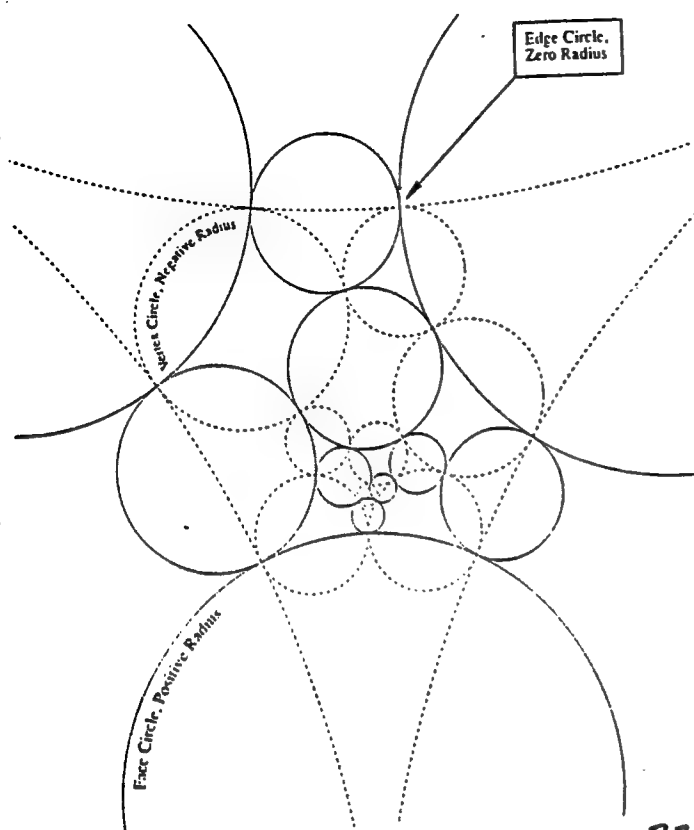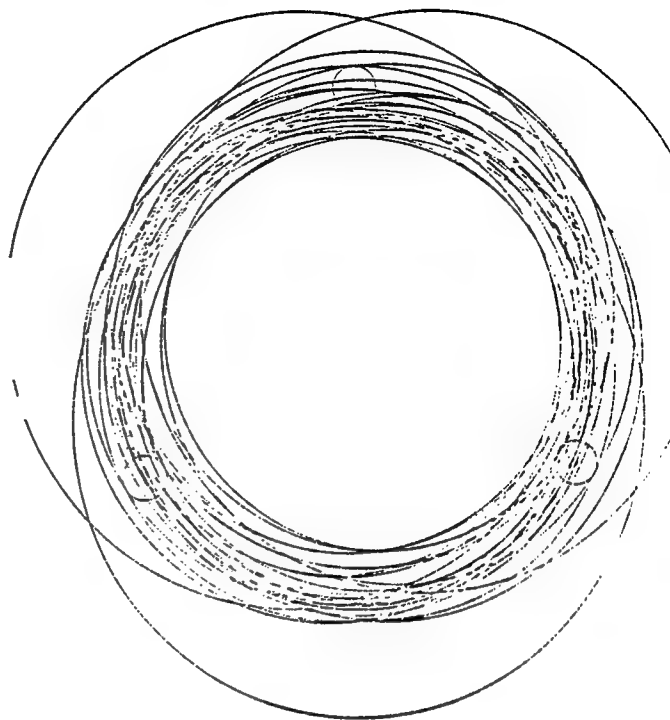
A Conjecture of Tutte...

Let G be a 3-connected planar graph. Can one properly draw G and its dual G* with straight line segments for edges, so that dual edges cross at 90°?

Yes!
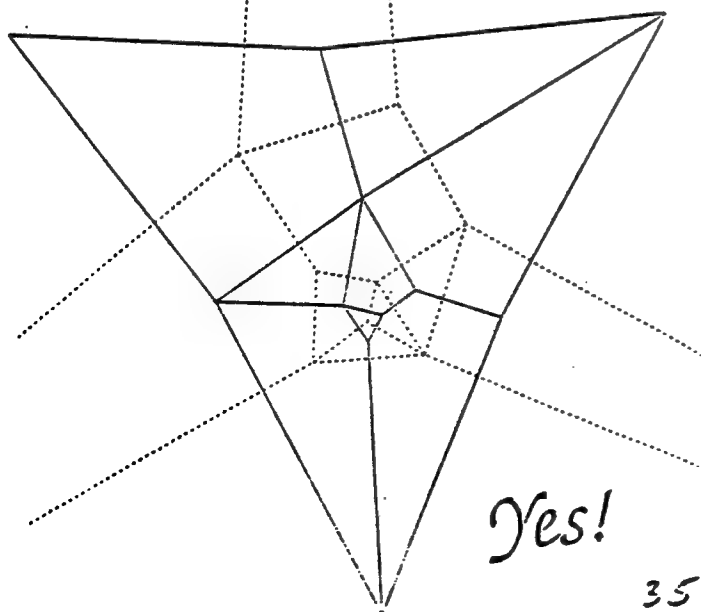
What about the unbounded faces?

For example...
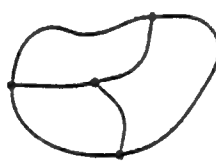
M
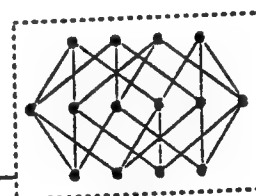
P$^+$(M)
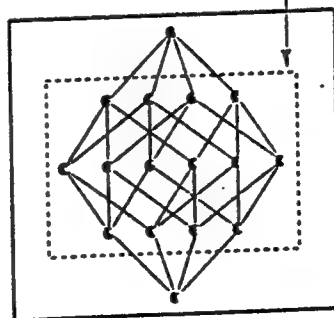
$2^{\{1,2,3,4\}}$ is not a circle order (Jamison), which implies P$^+$(M) is not a circle order.

but...3?

Theorem. If M is a 3-connected planar map, then $P^+(M)$ is a "cap order".
(Brightwell & Scheinerman)
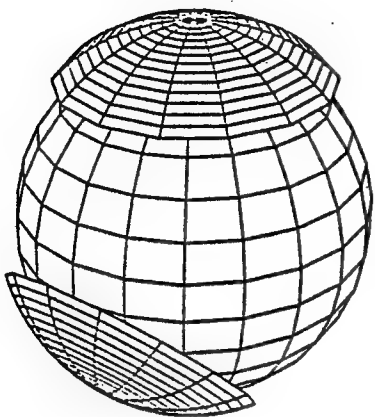
Note: $P^+(M)$ is the face lattice of a convex polyhedra in R^3.

37

"On Finding Transmitter-Receiver Matchings"

Jean R.S. Blair, University of Tennessee
Department of Computer Science, Knoxville, TN

On Finding

Transmitter - Receiver Matchings

Jean R.S. Blair[†]
University of Tennessee

and

S.S. Ravi[‡]
SUNY at Albany

1

Transmitter - Receiver Matching
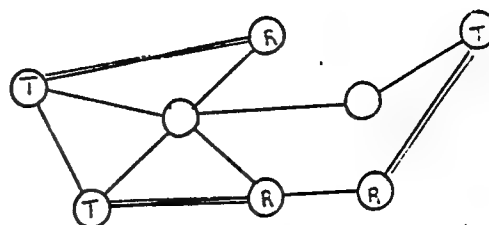
Given: a communication network with bidirectional links

Goal: to simultaneously test as many links as possible

Constraints:

(1) a transmitting site sends signals along all of the links emanating from it

(2) a site cannot be both a transmitter and a receiver at the same time

(3) two or more signals reaching a site at the same time interfere

2

Known Results for different graph structures:

NP-Complete for:

→ Graphs with max. degree 3 [Stockmyer, Vazirani]

→ Bipartite Graphs [Even, Goldreich, Moran, Tong]

→ Chordal Graphs [this paper]

Linear Time Algorithms for:

→ Trees [Farley, Proskurowski]

→ 2-Trees [Colbum, Proskurowski]

→ AC-Graphs [this paper]

Quadratic Time Algorithm for:

→ Interval Graphs [Spinrad]

3

AC Graphs

Graph G:
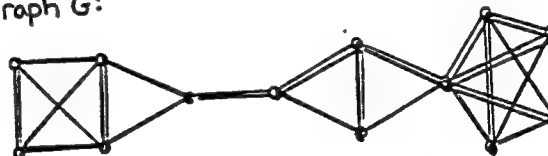
Clique Graph G_c
→ one node for each maximal clique in G
→ an edge between 2 nodes iff the corresponding cliques intersect
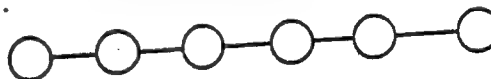
AC Graphs
→ graphs whose clique graphs are acyclic

Terminology
→ leaf node of G_c, leaf clique of G, parent clique, simplicial node of G

4

An AC Graph
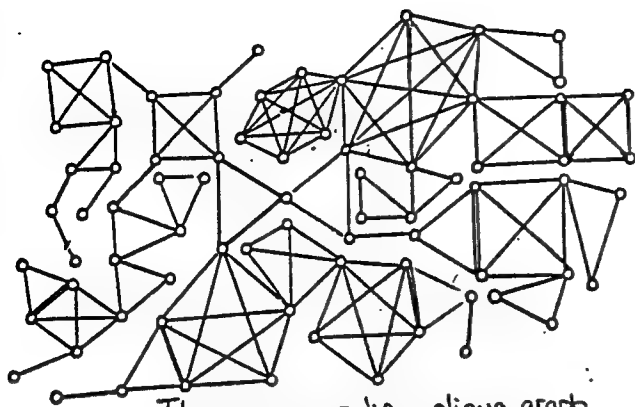
The corresponding clique graph
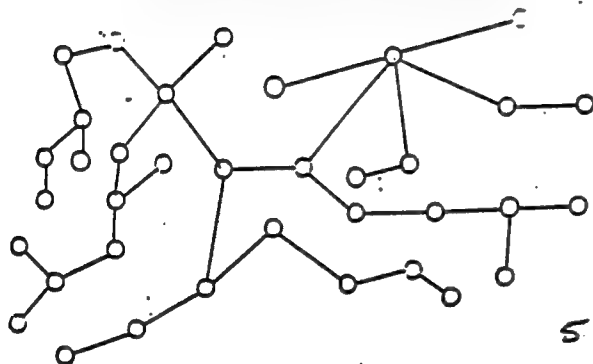
Some graphs that are not AC graphs.

G:

G_c:

H:

H_c:
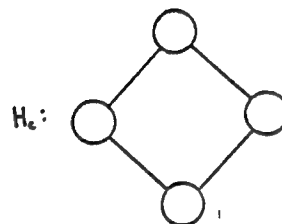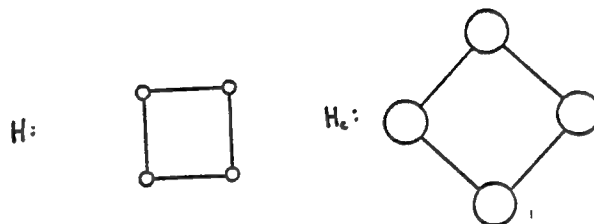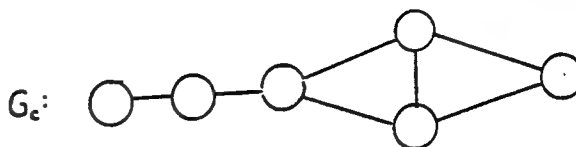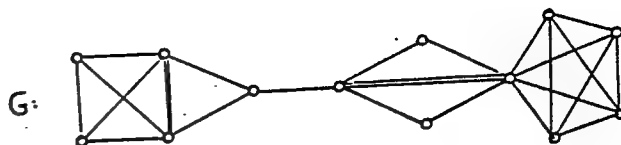
Useful Propertys of AC Graphs

Lemma 2.1 -- Any node of an AC Graph, G, belongs to at most two cliques.

Lemma 2.2 -- Any induced subgraph of an AC Graph is also an AC Graph.

Lemma 2.3 -- Every leaf clique contains at least one simplicial node.

Lemma 2.4 -- Every AC Graph is chordal.

Conflicts

TR-pairs : (x, y) and (w, z) conflict iff at least one of the edges $\{x, z\}$ and $\{y, w\}$ is in G
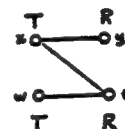
interference at ϵ

Edges : $\{x, y\}$ and $\{w, z\}$ conflict iff every orientation of the edges results in a c

Lemma 2.5 -- Two distinct edges $\{x, y\}$ and $\{w, z\}$ conflict iff the subgraph of G induce on $\{x, y, w, z\}$ contains a 3-cycle
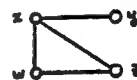
Orientability

A matching M is orientable iff there exists an orientation of M with no conflicts.

M -- a matching

V_M -- the nodes that comprise M

G_M -- the subgraph of G induced on V_M

Lemma 2.6 -- M is orientable iff G_M does not contain a cycle that uses an edge in M.

black edges belong to M

Lemma 2.7 -- If no pair of edges in M conflict, then M is orientable.

(Both results require that G be chordal).

9

Nodes in a Clique and Orientable Matchings

Lemma 3.1 -- Let $I = C_i \cap V_M$ for clique C_i and orientable matching M. Exactly one of following is true:
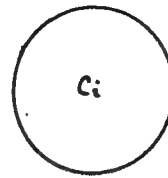
(a) $I = \emptyset$
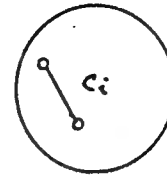
(b) for $x, y \in I$ $\{x,y\} \in M$ and $I - \{x,y\} = \emptyset$
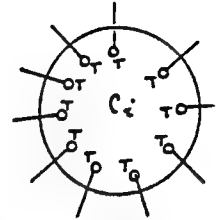
(c) nodes in I single endpoints pairs in M & in valid orientation labels are same

Lemma 3.3 -- If for $x, y \in C_i$ $\not\exists \{x,y\} \in M$ then at most one node from each $C_i \cap C_j$ $j \neq i$, can be in

10

Sketch of the Algorithm

With each clique C_i store PossiblePair(C_i)
(initial value is TRUE)

{Find a maximum cardinality orientable matching}

Until all nodes have been removed from G

(1) choose a leaf clique C_ℓ

(2) if PossiblePair(C_j)

choose a "good" pair from C_ℓ

remove from G all nodes that would conflict with the pair

set PossiblePair(C_p) to FALSE, if necessary

else remove C_ℓ from G

Orient the edges

11

Choosing a "good pair"

Lemma 3.1(b) → no matter what pair, no other nodes in C_g can belong to V_M

Lemma 3.1 → if choose any node from $C_\ell \cap C_p$ then no pairs come from C_p

→ if choose two nodes from $C_\ell \cap C_p$ then no other nodes come from C_p

Stay away from C_p as much as possible

Lemma 2.3 → $\exists$ at least one simplicial node in C_ℓ

12

Example — PossiblePair(C_p) = TRUE
C_ℓ contains >1 simplicial node
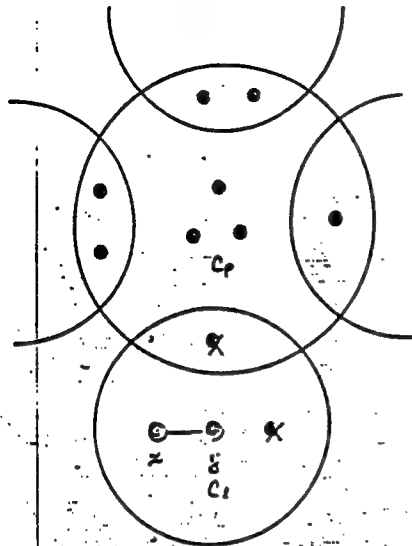
choose <x,y> and prune G

13

Example — PossiblePair(C_ℓ) = TRUE
C_ℓ contains one simplicial node
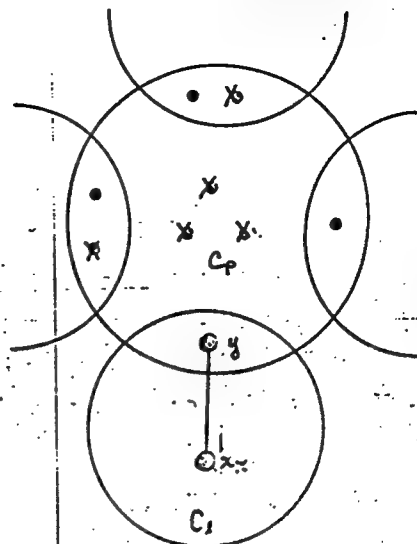
Choose <x,y> and prune G   (Lemma 3.3)

PossiblePair(C_p) ← FALSE

14

Alg TRM

1. Construct $G_c(V_c, E_c)$ (the clique graph of G)
2. $M \leftarrow \emptyset$
3. for each clique C_i do PossiblePair(C_i) ← true
4. while G_c contains two or more nodes do
   a. $c_l \leftarrow$ a leaf node of G_c
   b. $C_l \leftarrow$ the corresponding leaf clique of G
   c. $C_p \leftarrow$ the clique that intersects C_l
   d. if PossiblePair(C_l) then
      i. $x \leftarrow$ a simplicial node in C_l
      ii. If $C_l - \{x\}$ has a simplicial node then
          $y \leftarrow$ a simplicial node in $C_l - \{x\}$
          $P \leftarrow \emptyset$
          else
          $y \leftarrow$ a node in $C_l - \{x\}$
          if PossiblePair(C_p) then
          PossiblePair(C_p) ← false
          $P = \{$ all nodes in C_p except one from
                  each $C_i \cap C_p$, $i \neq l$ and $i \neq p\}$
          else $P \leftarrow \emptyset$
      iii. Add $\{x,y\}$ to M
      iv. Delete $C_l \cup P$ from G and c_l from G_c
      v. If $|C_p - P| = 1$ then Delete c_p from G_c
      else
      vi. Delete the simplicial nodes of C_l from G
      vii. Delete c_l from G_c
5. If G is not empty then
   a. $x \leftarrow$ a simplicial node in G's remaining clique
   b. $y \leftarrow$ a simplicial node distinct from x in G
   c. Add $\{x,y\}$ to M
6. Orient the edges of M using AlgOrient

15

Orientation Step

Given the set M, and G_M
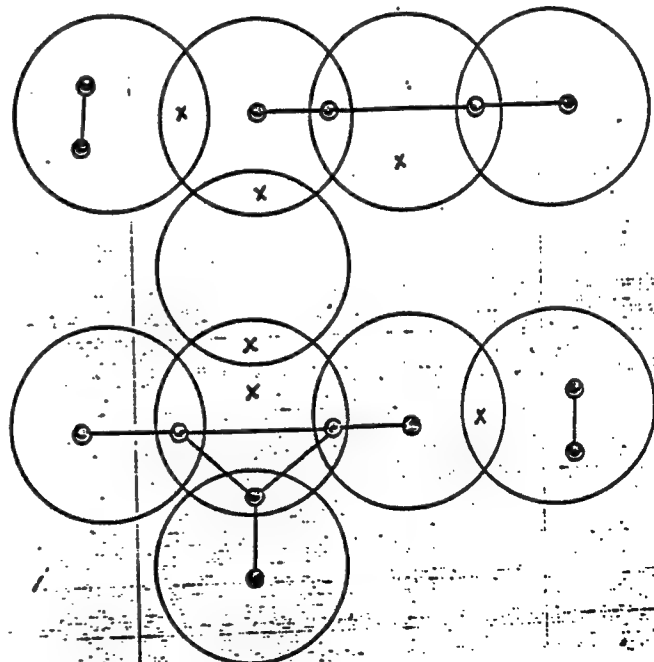
16

AlgOrient

(1) Find the connected components of G_M
(1a) Shrink non-matching portions of each component
(2) Perform Breadth-First-Search on each component labeling nodes on adjacent levels opposite.

---
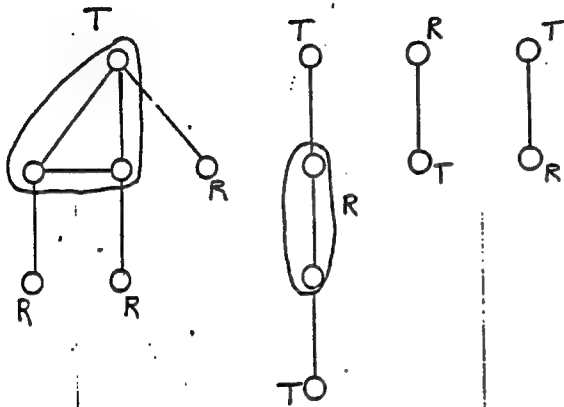
1. finding the set of maximal cliques -- standard techniques

2. constructing the clique graph -- use $\Theta(|c|^2+|v|)$ space to construct adjacency lists in $\Theta(|V|+|E|)$ time

5. for each leaf clique choose a pair and prune G and Ge

   • each clique is "the leaf clique" at most once

   • each clique is pruned as a parent clique at most once

   • time required to process a leaf clique (or a parent clique) is proportional to the number of nodes in the clique

   • since G is an AC-graph, each node of G belongs to at most two cliques

6. orient the edges in M -- similar to 2-coloring a bipartite graph

---

Final Remarks

TRM for Chordal Graphs is NP-Complete
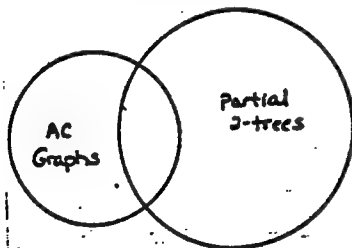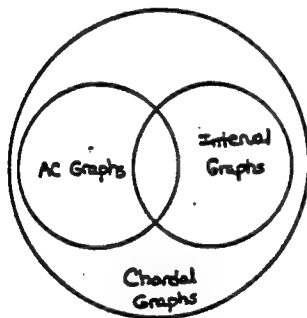
Determining if G is an AC-graph

Let G be a chordal graph.

Let G_c be its clique graph.

Property: If there is a cycle in G_c then there is a node in G that belongs to at least 3 cliques.

Lemma 2.1: Any node of an AC graph G belongs to at most 2 cliques.

> (1) check for chorality of G
>
> (2) find maximal cliques of G
>
> (3) check to see if any node in G belongs to more than 2 cliques

$\Theta(|V|+|E|)$ time

"The Jamison Method in Galois Geometries"

J. Chris Fisher, University of Regina, Canada
Visiting Clemson University, Department of Mathematical Sciences

JAMISON'S METHOD
Joint work with AIDEN BRUEN

$$\pi = AG(2,q) \quad \left\{ \begin{array}{l} \text{Affine plane} \\ \text{over } GF(q) \end{array} \right.$$

Definition.
"S BLOCKS THE LINES OF π":
For every line $\ell \in \pi$ there is a $P \in S$ with $P \in \ell$.

THEOREM (JAMISON, 1977)
A BLOCKING SET OF π MUST CONTAIN AT LEAST $2q-1$ points.

Cf. Projective planes:
$$|S'| = q+1 \qquad \begin{array}{l} \text{(since each pair} \\ \text{of lines meet)} \end{array}$$

Or (when S' contains no line)
$$|S'| = q + \sqrt{q} + 1$$

Facts needed for the proof:

① points: $(x,y) \longleftrightarrow z = x + iy \in GF(q^2)$
where $i \in GF(q^2) \backslash GF($
$\underbrace{\qquad}_{\text{elements of } GF(q)}$

② field automorphism of $GF(q^2)$ of period 2: $\bar{z} = z^q \quad (\text{recall } z^{q^2} = z)$

③ line: $\bar{z} + az + b = 0 \quad (= z^q + az + b)$

PROOF
suppose to the contrary that $|S| \leq 2q-2$

step 1
REPHRASE THE THEOREM AS A RELATIONSHIP INVOLVING SETS OF points in π.

original	dual
S contains $2q-2$ points	S^* has $2q-3$ affine lines $\cup \{\ell_\infty\}$
Every $\ell \in \pi$ contains at least one $P \in S$	Every $P \in \pi \backslash \{0\}$ lies on at least one $\ell \in S^*$

i.e.
$\boxed{S^* \text{ has } 2q-3 \text{ affine lines that cover the nonzero points of } \pi.}$

STEP 2
FORMULATE THE THEOREM IN TERMS OF POLYNOMIALS OVER $GF(q^2)$

Let S'^* consist of the lines
$$\ell_i = z^q + a_i z + b_i = 0 \quad b_i \neq 0$$
$$i = 1, \ldots, 2q-3$$

Assumption:
$$\pi \backslash \{0\} \subseteq S'^*$$

so $z^{q^2-1} - 1$ divides $\prod_{i=1}^{2q-3} \ell_i$

Or
$$\prod_{i=1}^{2q-3} (z^q + a_i z + b_i) = 0 \quad (\text{mod } z^{q^2-1} - 1)$$

STEP 3 CALCULATE

$(z^2+a_1z+b_1)(z^2+a_2z+b_2)\cdots(z^2+a_{2q-3}z+b_{2q-3})$

$=(z^2)^{2q-3}+\ldots+(b_1b_2\cdots b_{2q-3})\equiv 0 \pmod{z^{q^2-1}-1}$

key Observation:
The coefficient of $z^{q^2-1}=-\pi b_i \neq 0$

Try q z^2's and $q-3$ other factors
$\Rightarrow z^{q^2+?}$ Exponent is too big

Try $q-1$ z^2's and $q-2$ other factors.
$\Rightarrow z^{q^2-q}\times(\ )z^{q-2}=(\ )z^{q^2-2}$
Exponent is Too Small

$\Rightarrow C_{q^2-1}=0$
which is a contradiction ⊠

The Flock Theorem.

GIVEN AN ELLIPTIC QUADRIC IN PG(3,q) AND A SET OF q-1 DISJOINT CONICS PARTITIONING ALL BUT TWO OF ITS POINTS, THEN THE q-1 PLANES OF THOSE CONICS MUST CONTAIN A COMMON LINE THAT MISSES THE QUADRIC.

Blokhuis's Theorem.

FOR q ODD, IF A q-ELEMENT SUBSET OF GF(q²) CONTAINING 0 AND 1 HAS THE PROPERTY THAT THE DIFFERENCE OF ANY TWO OF ITS ELEMENTS IS A SQUARE OF GF(q²), THEN IT IS GF(q).

Defn
A flock is a set of conics that partition all but Two points of a sphere.
The flock is linear if the planes of the conics all contain a common line.

The FLOCK THEOREM (1973)
   J.A. THAS (for q even)
   W.F. Orr (for q odd)

Proof
Step 1 Rephrase.
Project the sphere (from one of the uncovered points) onto π, sending the other uncovered point to 0
The conics project to circles
$(z-a)\overline{(z-a)}=r$
Then for the q-1 circles to be disjoint they must have center 0 (i.e. a=0)

Step 2. Reformulate
Set $c_j = z^{q+1}-a_j z^q -a_j^q z -r_j$
$j=1,\ldots,q-1$

Claim:
$\prod_{j=1}^{q-1} c_j = z^{q^2-1}-1$ iff all $a_j=0$

step 3. Calculate.
$\prod c_j = \prod_{j\leq q-1}(z^{q+1}-a_j z^q) + \left\{\begin{array}{l}\text{Terms of degree}\\ \text{at most}\\ (q+1)(q-2)+1\end{array}\right\}$

Equate terms of degree $>(q+1)(q-2)+1$
$\prod_{j\leq q-1}[z^q(z-a_j)]=z^{q^2-1}$ iff all $a_j=0$
i.e. $\prod(z-a_j)=z^{q-1}$ iff all $a_j=0$

But $GF(q^2)[z]$ is a unique factorization domain ⊠

"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

Fred S. Roberts, Rutgers University
Dept. of Mathematics, Center of Operations Research (RUTCOR), and
Center for Discrete Mathematics and Theoretical Computer Science (DIMACS)
New Brunswick, NJ

ELEMENTARY, SUB-FIBONACCI, REGULAR, VAN LIER,
AND OTHER INTERESTING SEQUENCES

BY

FRED S. ROBERTS

DEPARTMENT OF MATHEMATICS
CENTER FOR OPERATIONS RESEARCH
AND
CENTER FOR DISCRETE MATHEMATICS AND THEORETICAL
COMPUTER SCIENCE
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ USA

MEASUREMENT THEORY:

THE THEORY OF MEASUREMENT IS CONCERNED WITH
UNDERSTANDING THE CONDITIONS UNDER WHICH
MEASUREMENT PROCESSES TAKE PLACE, WHAT KINDS OF
SCALES OF MEASUREMENT ONE GETS, AND WHAT KINDS OF
STATEMENTS WE CAN MAKE USING SCALES OF
MEASUREMENT.

MEASUREMENT THEORY AND COMBINATORICS:

IN THE PAST FEW YEARS, PROBLEMS OF UNIQUENESS
OF SCALES OF MEASUREMENT HAVE BEEN GIVING RISE TO
A VARIETY OF INTERESTING SEQUENCES OF POSITIVE
INTEGERS WITH FASCINATING COMBINATORIAL PROPERTIES.

THIS TALK:

IN THIS TALK, I MENTION SUCH SEQUENCES AND
DISCUSS THEIR COMBINATORIAL PROPERTIES. BECAUSE OF
THE SHORTNESS OF TIME, I CANNOT DESCRIBE THE
MEASUREMENT THEORY MOTIVATION EXCEPT IN ONE CASE
AND I SHALL CONCENTRATE ON JUST ONE COMBINATORIAL
PROBLEM: COUNTING THE NUMBER OF SEQUENCES OF
DIFFERENT KINDS.

REFERENCES

FISHBURN, MARCUS-ROBERTS, AND ROBERTS, "UNIQUE
FINITE DIFFERENCE MEASUREMENT," SIAM J. DISCR. MATH.,
1 (1988), 334-354.

FISHBURN AND ODLYZKO, "UNIQUE SUBJECTIVE
PROBABILITY ON FINITE SETS," J. RAMANUJAN MATH. SOC.,
4 (1989), 1-23.

FISHBURN, ODLYZKO, AND ROBERTS, "TWO-SIDED
GENERALIZED FIBONACCI SEQUENCES," FIBONACCI QUART.,
27 (1989), 352-361.

FISHBURN AND ROBERTS, "AXIOMS FOR UNIQUE SUBJECTIVE
PROBABILITY ON FINITE SETS," J. MATH. PSYCH., 33 (1989),
117-130.

FISHBURN AND ROBERTS, "ELEMENTARY SEQUENCES,
SUB-FIBONACCI SEQUENCES," TECHNICAL REPORT 90-51,
DIMACS CENTER.

FISHBURN AND ROBERTS, "UNIQUE FINITE CONJOINT
MEASUREMENT." MATH. SOC.SCI., 16 (1988), 107-143.

FISHBURN AND ROBERTS, "UNIQUENESS IN FINITE
MEASUREMENT," IN F.S. ROBERTS (ED.), APPLICATIONS OF
COMBINATORICS AND GRAPH THEORY TO THE BIOLOGICAL
AND SOCIAL SCIENCES, IMA VOL. 17, SPRINGER-VERLAG,
NEW YORK, 1989, 103-137. (SURVEY)

FISHBURN, ROBERTS, AND MARCUS-ROBERTS, "VAN LIER
SEQUENCES," DISCRETE APPL. MATH., 27 (1990), 209-220.

VAN LIER, L., "A SIMPLE SUFFICIENT CONDITION FOR THE
UNIQUE REPRESENTABILITY OF A FINITE QUALITATIVE
PROBABILITY BY A PROBABILITY MEASURE," J. MATH.
PSYCH., 33 (1989), 91-98.

THE FIBONACCI SEQUENCE

$F_1, F_2, \dots$

$F_1 = F_2 = 1$

$F_K = F_{K-1} + F_{K-2}, \quad K = 3, 4, \dots$

$F_1, F_2, \dots$ IS THE SEQUENCE $1, 1, 2, 3, 5, 8, 13, \dots$

ELEMENTARY SEQUENCES

MOTIVATION: "EXTENSIVE" MEASUREMENT
TERM INTRODUCED BY FISHBURN AND ROBERTS (1989).
VARIATION ON THE FIBONACCI SEQUENCE

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE

$X_1 = X_2 = 1$

$X_K > 1 \quad \longrightarrow \quad X_K = X_I + X_J, \quad$ SOME $I \neq J$.

EXAMPLES: $\quad 1, 1, 2, 3, 3, 5$

$\qquad\qquad 1, 1, 1, 2, 2, 4, 6$

$\mathcal{E}_N =$ COLLECTION OF ALL ELEMENTARY SEQUENCES OF LENGTH N

SUB-FIBONACCI SEQUENCES

TERM INTRODUCED BY FISHBURN AND ROBERTS (1989)

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE

$X_1 = X_2 = 1$

$X_K \leq X_{K-1} + X_{K-2}$

$\mathcal{S}_N =$ COLLECTION OF SUB-FIBONACCI SEQUENCES OF LENGTH N.

NOTE: $\mathcal{E}_N \subseteq \mathcal{S}_N$ FOR ALL N.

BY ENUMERATION, $\mathcal{E}_N = \mathcal{S}_N$ FOR $N \leq 6$.

THE SMALLEST SUB-FIBONACCI SEQUENCE WHICH IS NOT ELEMENTARY HAS LENGTH 7: $\quad 1, 1, 2, 2, 4, 4, 7$

THIS IS NOT ELEMENTARY BECAUSE 7 IS NOT THE SUM OF ANY TWO PRECEDING TERMS. ($\mathcal{S}_7$ HAS SEVEN SEQUENCES NOT IN $\mathcal{E}_7$.)

THEOREM (FISHBURN AND ROBERTS 1990):
$$|\mathcal{S}_N| = \sigma^{N^2(1+o(1))/2}$$

WHERE

$\sigma = (1+\sqrt{5})/2 = 1.61803 \quad$ (THE GOLDEN SECTION)

AND $o(1)$ IS A FUNCTION OF N THAT APPROACHES 0 AS N APPROACHES ∞.

COROLLARY: THE SAME ESTIMATE HOLDS FOR $|\mathcal{S}'_N|$, WHERE $\mathcal{S}'_N$ IS THE SUBSET OF $\mathcal{S}_N$ WHOSE ELEMENTS STRICTLY INCREASE FROM $K = 2$ ON.

OPEN QUESTION: FIND A SIMILAR RESULT FOR $|\mathcal{E}_N|$. THE FOLLOWING IS KNOWN:

THEOREM (FISHBURN AND ROBERTS 1990): $|\mathcal{E}_N|/|\mathcal{S}'_N| \longrightarrow 0$ AND HENCE $|\mathcal{E}_N|/|\mathcal{S}_N| \longrightarrow 0$ AS $N \longrightarrow \infty$.

SOME COUNTS:

N:	3	4	5	6	7	8	9	10	11
$|\mathcal{E}_N|$	2	4	10	31	120	578	3422	24504	205744
$|\mathcal{S}_N|$	2	4	10	31	127	711	3521	64349	1067509
$|\mathcal{S}'_N|$	1	1	2	6	27	177	1693	23039	455352

SO $|\mathcal{S}'_N|$ DOES NOT EXCEED $|\mathcal{E}_N|$ UNTIL $N = 11$.

REGULAR SEQUENCES

MOTIVATION: "SUBJECTIVE PROBABILITY" MEASUREMENT
INTRODUCED BY: FISHBURN AND ODLYZKO 1989

$X_1, X_2, \dots$

POSITIVE, NONDECREASING INTEGER SEQUENCE

$X_1 = X_2 = 1$

$X_J \leq \sum_{I=1}^{J-1} X_I \quad J = 3, 4, \dots$

EXAMPLES:

$N = 2$: $\quad 1, 1$

$N = 3$: $\quad 1, 1, 1; \quad 1, 1, 2$

$N = 4$: $\quad 1, 1, 1, 1; \quad\quad 1, 1, 1, 2; \quad\quad 1, 1, 1, 3;$

$\qquad\qquad 1, 1, 2, 2; \quad\quad 1, 1, 2, 3; \quad\quad 1, 1, 2, 4$

$\mathcal{R}_N =$ THE COLLECTION OF REGULAR SEQUENCES OF LENGTH N.

THEOREM (FISHBURN AND ODLYZKO 1989):
$$|\mathcal{R}_N| = 2^{N^2(1+o(1))/2}$$

SOME COUNTS

N:	2	3	4	5	6	7	8
$|\mathcal{R}_N|$	1	2	6	27	192	2260	47097

THE FOLLOWING THEOREM WILL BE USEFUL LATER.

THEOREM (FISHBURN AND ODLYZKO 1989): SUPPOSE X_1, X_2, ... IS A POSITIVE, NONDECREASING INTEGER SEQUENCE WITH $X_1 = X_2 = 1$. THEN X_1, X_2, ... IS A REGULAR SEQUENCE IF AND ONLY IF FOR EVERY J, THERE IS A SET $S \subseteq \{1, 2, ..., N\} - J$ SUCH THAT

(+) $$X_J = \sum_{I \in S} X_I.$$

FOR EXAMPLE, IN 1, 1, 2, 3:
2 IS $1 + 1$ AND 3 IS $1 + 2$.

IT IS CLEAR THAT CONDITION (+) IMPLIES REGULARITY SINCE IT IMPLIES THAT $X_J \leq \sum_{I=1}^{J-1} X_I$. THE CONVERSE IS HARDER.

VAN LIER SEQUENCES

MOTIVATION: "SUBJECTIVE PROBABILITY" MEASUREMENT TERM INTRODUCED BY VAN LIER (1989), FISHBURN AND ROBERTS (1989), FISHBURN, ROBERTS, AND MARCUS-ROBERTS (1990)

X_1, X_2, ...

POSITIVE, NONDECREASING INTEGER SEQUENCE

$X_1 = X_2 = 1$
$X_J \leq \sum_{I=1}^{J-1} X_I \quad J = 3, 4, ...$ (REGULAR SEQUENCE)
AND
$\forall J < K \leq N, \ \exists \ A \subseteq \{1, 2, ..., N\}$ S.T. $J \notin A$ AND
$X_K - X_J = \sum_{I \in A} X_I.$

EXAMPLE: 1, 1, 2, 4

$4 - 2 = 1 + 1$
$4 - 1 = 1 + 2$
$2 - 1 = 1$

EXAMPLE: 1, 1, 2, 3, 5, 8 $\quad (= F_1, F_2, F_3, F_4, F_5, F_6)$

$8 - 5 = 3$, $\ 8 - 3 = 5$, $\ 8 - 2 = 1 + 5$, $\ 8 - 1 = 2 + 5$, $5 - 3 = 2$, ETC.

IT IS EASY TO SEE THAT EVERY INITIAL SUBSEQUENCE OF THE FIBONACCI SEQUENCE IS VAN LIER.

THE FOLLOWING RESULT IS QUITE A BIT HARDER:

THEOREM (FISHBURN, ROBERTS, & MARCUS-ROBERTS 1990): $\mathcal{F}_N \subseteq \mathcal{V}_N$, I.E., EVERY SUB-FIBONACCI SEQUENCE IS VAN LIER.

NOT EVERY REGULAR SEQUENCE IS VAN LIER. THE SMALLEST REGULAR SEQUENCE WHICH IS NOT VAN LIER IS 1, 1, 2, 4, 5.
$5 - 2$ IS NOT A SUM OF TERMS FROM $\{1, 1, 4\}$.

$\mathcal{V}_N$ = THE COLLECTION OF VAN LIER SEQUENCES OF LENGTH N.

OPEN QUESTION: FIND AN ASYMPTOTIC FORMULA FOR $|\mathcal{V}_N|$.

CONJECTURE (FISHBURN AND ROBERTS 1989): $|\mathcal{V}_N|/|\mathcal{F}_N| \longrightarrow 0$ AS $N \longrightarrow \infty$.

WE HAVE SHOWN THAT $|\mathcal{V}_N|/|\mathcal{F}_N| \leq \lambda$ FOR SOME CONSTANT $\lambda < 1$ ($\lambda = 0.9$ SUFFICES).

SOME COUNTS

N	2	3	4	5	6	7	8
$|\mathcal{V}_N|$	1	2	6	26	164	1529	21439

ASIDE: DISTINCTION BETWEEN REGULAR AND VAN LIER SEQUENCES

A BASIC STRUCTURAL PROPERTY THAT SEPARATES VAN LIER SEQUENCES FROM REGULAR SEQUENCES IS CALLED A GAP.

SUPPOSE X_1, X_2, ..., X_N IS A REGULAR SEQUENCE. WE SAY THAT IT HAS A GAP AT X_J IF

$$X_{J+1} > \sum_{I=1}^{J-1} X_I + 1.$$

FOR INSTANCE, CONSIDER 1, 1, 2, 3, 6.

THERE IS A GAP AT $X_4 = 3$ BECAUSE $6 > (1 + 1 + 2) + 1$.

THEOREM (FISHBURN, ROBERTS, AND MARCUS-ROBERTS 1990): EVERY REGULAR SEQUENCE WITHOUT GAPS IS A VAN LIER SEQUENCE.

THEOREM (FISHBURN, ROBERTS, AND MARCUS-ROBERTS
1990): SUPPOSE $X_1, X_2, ..., X_N$ IS A VAN LIER SEQUENCE.
THEN

(1) EVERY ONE-TERM EXTENSION OF $X_1, X_2, ..., X_N$ TO
A REGULAR SEQUENCE IS VAN LIER IF AND ONLY IF X_1,
$X_2, ..., X_N$ HAS NO GAPS.

(2) IF $X_1, X_2, ..., X_N$ HAS A GAP AT X_J AND

$$Y = X_J + T + \sum_{I=J+2}^{N} X_I$$

WITH

$$\sum_{I=1}^{J-1} X_I < T < X_{J+1}.$$

THEN $X_1, X_2, ..., X_N, Y$ IS A ONE-TERM REGULAR
EXTENSION WHICH IS NOT VAN LIER.

EXAMPLE: 1, 1, 2, 3, 6 IS VAN LIER AND THERE IS A GAP
AT $X_J = 3$. THEN $\sum_{I=J+2}^{N} X_I = 0$. TAKE $T = 5$ AND
$Y = X_J + T + \sum_{I=J+2}^{N} X_I = 3 + 5 + 0 = 8$. THE SEQUENCE
1, 1, 2, 3, 6, 8 IS REGULAR BUT NOT VAN LIER. (NOTE
THAT 8 - 3 IS NOT A SUM OF OTHER TERMS.)

NOTATION: A . B MEANS THAT NOT A > B AND NOT
B > A, I.E., A AND B ARE JUDGED SUBJECTIVELY
EQUALLY LIKELY.

EXAMPLE 1: SUPPOSE N = 4 AND > IS DEFINED BY
THE FOLLOWING:

$$\{3\} . \{1,2\}$$
$$\{4\} . \{1,3\}$$
$$\{2,3\} . \{1,4\}$$

AN AGREEING PROBABILITY MEASURE IS GIVEN BY

(*) $P(\{1\}) = 1/10, P(\{2\}) = 2/10, P(\{3\}) = 3/10, P(\{4\}) = 4/10,$

WITH THE REST OF P DEFINED BY FINITE ADDITIVITY.
THIS UNIQUELY AGREES BECAUSE IT IS THE UNIQUE
SOLUTION TO THE SYSTEM OF EQUATIONS

$$P(\{3\}) = P(\{1\}) + P(\{2\})$$
$$P(\{4\}) = P(\{1\}) + P(\{3\})$$
$$P(\{2\}) + P(\{3\}) = P(\{1\}) + P(\{4\})$$

ASIDE: CONNECTION TO MEASUREMENT THEORY

ONE OF THE MOST INTERESTING PROBLEMS IN THE
THEORY OF MEASUREMENT CONCERNS SUBJECTIVE
JUDGEMENTS ABOUT PROBABILITIES. LET $\mathcal{A}_N$ BE THE
SET OF ELEMENTS OF THE FINITE BOOLEAN ALGEBRA
CONSISTING OF ALL SUBSETS OF $\{1, 2, ..., N\}$. SET $\{I\}$
IS CALLED AN ATOM OF $\mathcal{A}_N$.

LET > BE A BINARY RELATION ON $\mathcal{A}_N$, WITH A > B
INTERPRETED TO MEAN THAT A IS JUDGED
SUBJECTIVELY MORE PROBABLE THAN B. WE SAY A
(FINITELY ADDITIVE) PROBABILITY MEASURE P ON $\mathcal{A}_N$
AGREES WITH > IF

$$A > B \quad \text{IFF} \quad P(A) > P(B)$$

FOR ALL A, B IN $\mathcal{A}_N$. IT IS A VERY OLD QUESTION OF
MEASUREMENT THEORY TO UNDERSTAND CONDITIONS ON
THE BINARY RELATION $(\mathcal{A}_N, >)$ UNDER WHICH IT
AGREES WITH SOME PROBABILITY MEASURE. THE
MEASURE IS SAID TO AGREE UNIQUELY IF IT IS THE ONLY
AGREEING PROBABILITY MEASURE.

EXAMPLE 2: SUPPOSE N = 3 AND P IS DEFINED BY

$$\{3\} . \{1,2\}.$$

THEN ONE AGREEING PROBABILITY MEASURE IS GIVEN BY

$$P(\{1\}) = 1/6, P(\{2\}) = 2/6, P(\{3\}) = 3/6.$$

BUT THIS IS NOT UNIQUE, SINCE A SECOND AGREEING
PROBABILITY MEASURE IS GIVEN BY

$$P(\{1\}) = 2/10, P(\{2\}) = 3/10, P(\{3\}) = 5/10.$$

EXAMPLE 3. SUPPOSE N = 2 AND P IS DEFINED BY

$$\{1\} > \{2\}.$$

THEN THERE ARE INFINITELY MANY AGREEING
PROBABILITY MEASURES P, WITH

$$P(\{1\}) = a, \quad P(\{2\}) = 1-a,$$

FOR ANY a WITH $1 > a > 1/2$.

LET US TAKE THE UNIQUE SOLUTION (*) IN THE FIRST EXAMPLE. WE CAN TRANSLATE THIS INTO A SEQUENCE OF POSITIVE INTEGERS 1, 2, 3, 4 (WITH NO COMMON DIVISOR) BY MULTIPLYING BY THE DENOMINATOR 10. CONVERSELY, ANY FINITE SEQUENCE OF POSITIVE INTEGERS CAN BE THOUGHT OF AS A SEQUENCE OF PROBABILITIES BY NORMALIZING, I.E., BY DIVIDING EACH ELEMENT BY THE SUM OF ELEMENTS IN THE SEQUENCE. LET US CALL A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR WHICH ARISES FROM A BINARY RELATION > BY FINDING A UNIQUELY AGREEING PROBABILITY MEASURE A UNIQUE PROBABILITY SEQUENCE. THUS, 1, 2, 3, 4 IS A UNIQUE PROBABILITY SEQUENCE WHILE 1, 2, 3 IS NOT.

FISHBURN AND ODLYZKO [1989] PROVE THAT ALL REGULAR SEQUENCES ARE UNIQUE PROBABILITY SEQUENCES. HOWEVER, NOT ALL UNIQUE PROBABILITY SEQUENCES ARE REGULAR. FOR INSTANCE, 1, 2, 3, 4 IS NOT REGULAR: WE DO NOT HAVE $X_2 = 1$.

THEOREM (FISHBURN AND ODLYZKO 1989): A NONDECREASING SEQUENCE $X_1, X_2, ..., X_N$ OF POSITIVE INTEGERS WITH NO COMMON DIVISOR IS A UNIQUE PROBABILITY SEQUENCE IF AND ONLY IF IT IS THE SOLUTION TO N-1 LINEARLY INDEPENDENT EQUATIONS OF THE FORM

$$\sum_{I \in S} X_1 = \sum_{J \in T} X_J,$$

WHERE $S, T \subseteq \{1, 2, ...N\}$ AND $S \cap T = \phi$.

FOR INSTANCE, THE SEQUENCE 1, 2, 3, 4 IS A UNIQUE PROBABILITY SEQUENCE BECAUSE IT IS THE SOLUTION TO THE $4 - 1 = 3$ LINEARLY INDEPENDENT EQUATIONS

$$X_3 = X_1 + X_2$$
$$X_4 = X_1 + X_3$$
$$X_2 + X_3 = X_1 + X_4$$

ALSO, 1, 2, 2, 3 IS A UNIQUE PROBABILITY SEQUENCE SINCE IT IS THE SOLUTION TO THE $4 - 1$ LINEARLY INDEPENDENT EQUATIONS

$$X_2 = X_3$$

THIS CORRESPONDS TO THE SUBJECTIVE PROBABILITY CONSTRAINTS

$$P(\{2\}) = P(\{3\})$$
$$P(\{1,2\}) = P(\{4\})$$
$$P(\{1,4\}) = P(\{2,3\})$$

THIS SEQUENCE 1, 2, 2, 3 IS AGAIN IRREGULAR SINCE $X_2 \neq 1$. IT TURNS OUT THAT 1, 2, 3, 4 AND 1, 2, 2, 3 ARE THE ONLY IRREGULAR UNIQUE PROBABILITY SEQUENCES OF LENGTH 4.

HOWEVER, THERE ARE 75 IRREGULAR UNIQUE PROBABILITY SEQUENCES OF LENGTH 5, INCLUDING 1, 1, 3, 3, 5 AND 2, 2, 2, 3, 3. THE FORMER IS INTERESTING. IT IS A UNIQUE PROBABILITY SEQUENCE SINCE IT IS THE SOLUTION TO THE $5 - 1 = 4$ LINEARLY INDEPENDENT EQUATIONS

$$X_1 = X_2$$
$$X_3 = X_4$$
$$X_1 + X_2 + X_3 = X_5$$
$$X_3 + X_4 = X_1 + X_5.$$

IT IS NOT REGULAR SINCE 3 IS NOT LESS THAN OR EQUAL TO THE SUM OF THE PREVIOUS TERMS IN THE SEQUENCE, $1 + 1$.

LET $\mathcal{A}_N$ BE THE COLLECTION OF UNIQUE PROBABILITY SEQUENCES OF LENGTH N. RECALL THAT $\mathcal{R}_N$ IS THE COLLECTION OF REGULAR SEQUENCES OF LENGTH N.

THEOREM (FISHBURN AND ODLYZKO 1989): $|\mathcal{R}_N|/|\mathcal{A}_N| \to 0$ AS $N \to \infty$.

WE DO NOT KNOW MUCH ABOUT $|\mathcal{A}_N|$. HOWEVER, WE HAVE THE FOLLOWING UPPER BOUND:

THEOREM (FISHBURN AND ODLYZKO): $|\mathcal{A}_N| \leq 3^{N^2(1+o(1))}$.

WHERE DO REGULAR AND VAN LIER SEQUENCES ARISE?

THE PROBLEM OF FINDING CONDITIONS UNDER WHICH THERE IS A FINITELY ADDITIVE PROBABILITY MEASURE WHICH AGREES WITH A GIVEN BINARY RELATION "SUBJECTIVELY MORE PROBABLE THAN" $(\mathcal{A}_N, >)$ IS AN OLD PROBLEM. SOME NECESSARY CONDITIONS WERE STATED BY BRUNO DE FINETTI IN 1931. DEFINE $A \geq B$ TO MEAN THAT EITHER $A > B$ OR $A \sim B$.

DE FINETTI AXIOMS

AXIOM A1. $\geq$ IS TRANSITIVE AND COMPLETE ($A \geq B$ OR $B \geq A$ FOR ALL A, B IN $\mathcal{A}_N$)

AXIOM A2. $\{1, 2, ..., N\} > \phi$

AXIOM A3. $A \geq \phi$

Axiom A4. If $(A \cup B) \cap C = \phi$, THEN

$$A \geq B \quad \text{IFF} \quad A \cup C \geq B \cup C$$

IT IS EASY TO SEE THAT THESE FOUR AXIOMS ARE NECESSARY FOR THE EXISTENCE OF AN AGREEING PROBABILITY MEASURE. IT WAS SHOWN BY KRAFT, PRATT, AND SEIDENBERG IN 1959 THAT THEY ARE NOT SUFFICIENT.

VARIOUS CONDITIONS CAN BE ADDED TO THESE AXIOMS TO GIVE SUFFICIENT CONDITIONS. ONE SIMPLE CONDITION WAS ADDED BY KRAFT, PRATT, AND SEIDENBERG [1959]. IT SAYS THAT IF $A_1, A_2, ..., A_M, B_1, B_2, ..., B_M$ ARE CHOSEN FROM $\mathcal{A}_N$, IF EVERY ATOM IS INCLUDED IN AS MANY A_J AS B_J, AND IF $A_J \geq B_J$ FOR $J = 1, 2, ..., M-1$, THEN $B_M \geq A_M$.

THE NEXT QUESTION IS: UNDER WHAT CONDITIONS IS THERE A UNIQUELY AGREEING PROBABILITY MEASURE. STARTING WITH THE DE FINETTI AXIOMS? RATHER COMPLICATED CONDITIONS FOR UNIQUE AGREEMENT WERE GIVEN BY LUCE [1967] AND BY ROBERTS [1979]. THE FOLLOWING MUCH SIMPLER AXIOM WAS GIVEN BY FISHBURN AND ROBERTS [1989].

AXIOM U1: SUPPOSE X IS AN ATOM SUCH THAT $X > Y > \phi$ FOR SOME ATOM Y. THEN THERE IS AN EVENT $A(X)$ IN $\mathcal{A}_N$ SO THAT $X \cdot A(X)$ AND $X > Y$ FOR EVERY ATOM Y IN $A(X)$.

(PUT ANOTHER WAY, THE CONCLUSION OF THIS AXIOM SAYS THAT THERE ARE ATOMS $Y_1, Y_2, ..., Y_K$ SO THAT $X \cdot Y_1 \cup Y_2 \cup ... Y_K$ AND $X > Y_I$, $I = 1, 2, ..., K$.)

THIS IS RELATED TO THE FISHBURN-ODLYZKO CHARACTERIZATION OF REGULAR SEQUENCES AS NONDECREASING SEQUENCES OF POSITIVE INTEGERS SUCH THAT $X_1 = X_2 = 1$ AND SUCH THAT EACH X_J IS A SUM OF OTHER X_I, $I \neq J$.

THEOREM (FISHBURN AND ROBERTS 1989): GIVEN $(\mathcal{A}_N, >)$, THE DE FINETTI AXIOMS PLUS AXIOM U1 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHBURN AND ROBERTS 1989): A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR DEFINES A REGULAR SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION $(\mathcal{A}_N, >)$ WHICH SATISFIES AXIOM U1.

ANOTHER AXIOM IS DUE TO VAN LIER [1989].

AXIOM U2. FOR EVERY I, $J \in \{1, 2, ..., N\}$, IF $\{I\} > \{J\}$, THERE IS $C \in \mathcal{A}_N$ SUCH THAT $\{I\} \cdot \{J\} \cup C$.

THEOREM (VAN LIER 1989): GIVEN $(\mathcal{A}_N, >)$, THE DE FINETTI AXIOMS PLUS AXIOM U2 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHBURN AND ROBERTS 1989): A REGULAR SEQUENCE IS A VAN LIER SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION $(\mathcal{A}_N, >)$ WHICH SATISFIES AXIOM U2.

(TYPE A) TWO-SIDED GENERALIZED FIBONACCI SEQUENCES

MOTIVATION: "DIFFERENCE" MEASUREMENT INTRODUCED BY FISHBURN, MARCUS-ROBERTS, AND ROBERTS (1988) AND FISHBURN, ODLYZKO, AND ROBERTS (1989).

THIS IS A SEQUENCE OF POSITIVE INTEGERS WHICH, IN CONTRAST TO ALL THE TYPES OF SEQUENCES SO FAR, MAY BE DECREASING.

START WITH A PAIR OF ADJACENT 1'S. CONSTRUCT THE SEQUENCE INSIDE-OUT BY ADDING ONE TERM AT A TIME WHOSE VALUE IS A SUM OF ONE OR MORE CONTIGUOUS TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 5, 4, 1, 1, 2, 4

THIS IS BUILT UP AS:

    1, 1, 2
    4, 1, 1, 2
    5, 4, 1, 1, 2
    5, 4, 1, 1, 2, 4

a_N = THE COLLECTION OF (TYPE A) TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, AND ROBERTS 1988 AND FISHBURN, ODLYZKO, AND ROBERTS 1989):</u>
$$|a_N| = N^{N(1+o(1))}$$

IN FACT, FISHBURN, ODLYZKO AND ROBERTS SHOW THAT

$$a_N \sim \frac{K}{2} \sqrt{\frac{e}{\pi}} \frac{e^{2\sqrt{N}}(N-1)!}{N^{1/4}}.$$

WHERE

$$K = e^{-1} - \int_0^1 [\exp(\frac{1}{1-Y})/(1-Y)]dY = 0.145495...$$

---

<u>TYPE C TWO-SIDED GENERALIZED FIBONACCI SEQUENCES</u>

MOTIVATION: "DIFFERENCE" MEASUREMENT

SAME AS TYPE A TWO-SIDED GENERALIZED FIBONACCI SEQUENCES WITH THE NEW TERM BEING A SUM OF ONE OR MORE PREVIOUS TERMS, BUT NOT NECESSARILY OF CONTIGUOUS TERMS AND NOT NECESSARILY OF TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 3, 1, 1, 2, 6

THIS IS BUILT UP AS:
   1, 1, 2
   3, 1, 1, 2
   3, 1, 1, 2, 6

IT CANNOT BE BUILT UP BY ADDING CONTIGUOUS TERMS EACH TIME, SINCE 6 CAN ONLY BE OBTAINED AS 3 + 1 + 2.

γ_N = THE COLLECTION OF TYPE C TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, & ROBERTS 1988):</u>
$$|\gamma_N| = 2^{(N^2/2)(1+o(1))}$$

<u>TYPE B TWO-SIDED GENERALIZED FIBONACCI SEQUENCES</u>

MOTIVATION: "DIFFERENCE" MEASUREMENT

SAME AS TYPE A TWO-SIDED GENERALIZED FIBONACCI SEQUENCES WITH THE NEW TERM BEING A SUM OF ONE OR MORE CONTIGUOUS TERMS, BUT NOT NECESSARILY OF TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: 6, 1, 1, 2, 4

THIS IS BUILT UP AS:
   1, 1, 2
   1, 1, 2, 4
   6, 1, 1, 2, 4

THIS IS NOT ATTAINABLE IF WE INSIST THAT EACH NEW TERM IS A SUM OF TERMS IMMEDIATELY ADJACENT TO THE NEW ONE.

β_N = THE COLLECTION OF TYPE B TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

<u>THEOREM (FISHBURN, MARCUS-ROBERTS, AND ROBERTS 1988 AND FISHBURN, ODLYZKO, AND ROBERTS 1989):</u>
$$|\beta_N| = N^{2N(1+o(1))}$$

THE FOLLOWING COUNTS ARE KNOWN:

N	2	3	4	5	6
$|a_N|$	1	3	14	85	626
$|\beta_N|$	1	3	16	171	2433
$|\gamma_N|$	1	3	16	185	

ALL OTHER VALUES ARE STILL OPEN.

BIREGULAR SEQUENCES

MOTIVATION: "CONJOINT" MEASUREMENT
INTRODUCED BY FISHBURN AND ROBERTS (1988)

THESE ARE TWO-BLOCK SEQUENCES OF POSITIVE INTEGERS
$X_1, X_2, ..., X_M / Y_1, Y_2, ..., Y_N$

THEY ARE BUILT UP BY STARTING WITH 1 IN EACH
BLOCK AND ADDING ONE TERM AT A TIME (TO EITHER
BLOCK) THAT IS ADJACENT TO THE TERMS ALREADY
SPECIFIED FOR THE BLOCK AND WHOSE VALUE EQUALS A
SUM OF TERMS ALREADY SPECIFIED FOR THE OTHER
BLOCK.

EXAMPLE: 2, 3, 1, 1, 7 / 6, 1, 2, 10

BUILT UP AS:
    1 / 1
    1, 1 / 1
    1, 1 / 1, 2
    3, 1, 1 / 1, 2
    2, 3, 1, 1 / 1, 2
    2, 3, 1, 1 / 6, 1, 2
    2, 3, 1, 1, 7 / 6, 1, 2
    2, 3, 1, 1, 7 / 6, 1, 2, 10

$G_B(M,N)$ = THE COLLECTION OF BIREGULAR SEQUENCES OF
LENGTHS M AND N.

THEOREM (FISHBURN AND ROBERTS 1988):
$$|G_B(M,N)| = (2^N-1)^{M(1+o(1))}$$

INTERVAL-RESTRICTED BIREGULAR SEQUENCES

MOTIVATION: "CONJOINT" MEASUREMENT

SAME AS BIREGULAR SEQUENCES BUT ADD THE
REQUIREMENT THAT EACH TERM IS A SUM OF
CONTIGUOUS TERMS ALREADY SPECIFIED IN THE OTHER
BLOCK.

NOTE THAT THE LAST EXAMPLE WORKS UP UNTIL THE
LAST STEP. IN THE LAST STEP, 10 CANNOT BE ADDED.
HOWEVER, 14 COULD BE, GIVING US
        2, 3, 1, 1, 7 / 6, 1, 2, 14

$C_B(M,N)$ = THE COLLECTION OF INTERVAL-RESTRICTED
BIREGULAR SEQUENCES OF LENGTHS M AND N.

THEOREM (FISHBURN AND ROBERTS 1988):
$$|C_B(M,N)| = \binom{N+1}{2}^{M(1+o(1))}$$

NOTE THAT $|G_B(M,1)| = |C_B(M,1)| = 1$ FOR ALL M.

ONE CAN ALSO SEE THAT $|G_B(M,2)| = |C_B(M,2)|$ FOR
ALL M. THE FOLLOWING VALUES ARE KNOWN:

M	2	3	4	5	6	7	8
$|G_B(M,2)|$	5	19	80	243	841	2859	9573

$|G_B(9,2)|$ IS ALREADY NOT KNOWN.

"Generating k-element Subsets of an n-element Set"

Michael S. Jacobson, University of Louisville
Department of Mathematics, Louisville, KY

Generating k-element subsets of an n-element set with DeBruijn Graphs

M. S. Jacobson,
E. Kubicka,
G. Kubicki,
Department of Mathematics
University of Louisville
Louisville, KY 40292
and
A. Schwenk
Department of Mathematics and Statistics
Western Michigan University
Kalamazoo, Michigan 49008

Abstract

1

This tour is brought to you

by

the following problem:

Find an efficient* way to

generate all k-element

subsets of an n-element set.!!

·What does efficient mean?

2

Generate <u>all</u> subsets of an n-element set.

Binary representation of $0 - (2^n - 1)$

yields a "computer understandable"

way to generate these sets.

BUT

An *excessive* amount of work

for the computer to go from

$2^{n-1} - 1 = 011...1$ to $2^{n-1} = 100...0$

Is there a sequence which proceeds from subset to subset without many elements being exchanged?

3

Proceed thru the subsets with exactly one

"bit" changing, either 0 to 1, or 1 to 0.

One element difference from subset to subset.

GRAY CODES

0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	0	0
1	1	0	0
1	0	0	0
1	0	0	1
1	0	1	1
1	0	1	0
1	1	1	0
1	1	1	1
1	1	0	1
0	1	0	1
0	1	1	1

4

Utilize the power of the computer!!

Can we generate all binary sequences

of length n by a "shift" ??

$\underline{0}$ 0 1 0 1 1 1 0 1
$\underline{0}$ 1 0 1 1 1 0 1 0
$\underline{1}$ 0 1 1 1 0 1 0 1
$\underline{0}$ 1 1 1 0 1 0 1 0
$\underline{1}$ 1 1 0 1 0 1 0 1
$\underline{1}$ 1 0 1 0 1 0 1 1
$\underline{1}$ 0 1 0 1 0 1 1 1

. . .

Does there exist a sequence of length 2^n so that each sequence of length n occurs as a consecutive subsequence exactly once (wrapping allowed) ??

5

6

Efficient way to store all subsets of an n element set.

Efficient way to generate all subsets.

Efficient way to generate all subsets many times.

Efficient way to generate a <u>random</u> subset.

7
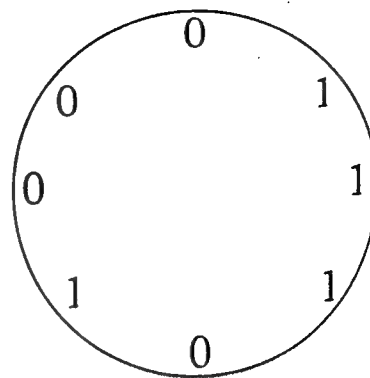
Combinatorial Tools
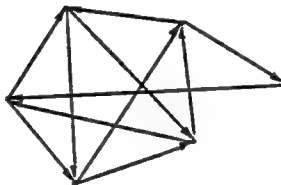
Directed Graph   Graph
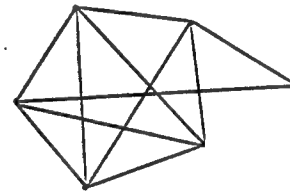
<u>Connected</u>

<u>Eulerian Digraph (Graph)</u>

   Starting at any point, trace thru the digraph (graph)

traversing each edge <u>exactly once</u>.

<u>Euler (1736) Good (1946)</u> If D is a connected digraph with

$id(x) = od(x)$ for all vertices x then D is Eulerian.

8

Do there exist <u>circular</u> sequences of length 2^n with each n sequence occurring exactly once?
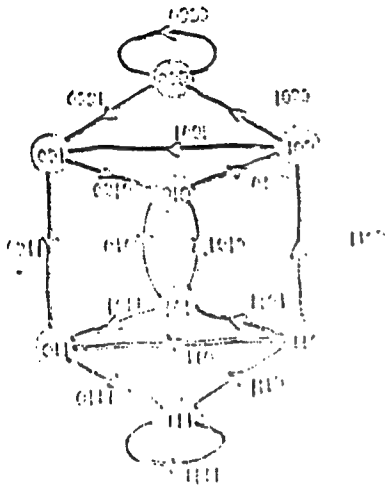
... and now a message from the sponsor

"But we only want the <u>k element</u> subsets!!??"

Generate all the subsets, and use only the ones you need.

Can we use shift registers ??

Double <u>Shift</u> <u>Registers</u>

```
110100101000
 010010100011
  001010001101
   101000110100
    100011010001
         ... ?
```

Does there exist a circular sequence of order

$2\binom{n}{k}$ so that by double shifting all k element

subsets are generated ?

Consider a different graph ...

00111
11100
10011
01110   01101
10110   10101
11001   11010
01011

14

Hamiltonian cycle $\longleftrightarrow$ Eulerian circuit

$\uparrow$

Connected $\longleftrightarrow$ Connected

n odd

$a_1 \ a_2 \ a_3 \ \ldots \ a_i \ a_{i+1} \ \ldots \ a_n$
$a_3 \ a_4 \ a_5 \ \ldots \ a_{i+3} \ \ldots \ a_2$

$\ldots$

$a_i \ a_{i+1} \ \ldots \ a_n \ a_1 \ \ldots \ a_{i-1}$
$a_{i+1} \ a_{i+2} \ \ldots \ a_{i-1} \ a_{i+1} \ a_i$

$\ldots$

$a_1 \ a_2 \ a_3 \ \ldots \ a_{i+1} \ a_i \ \ldots \ a_n$

15
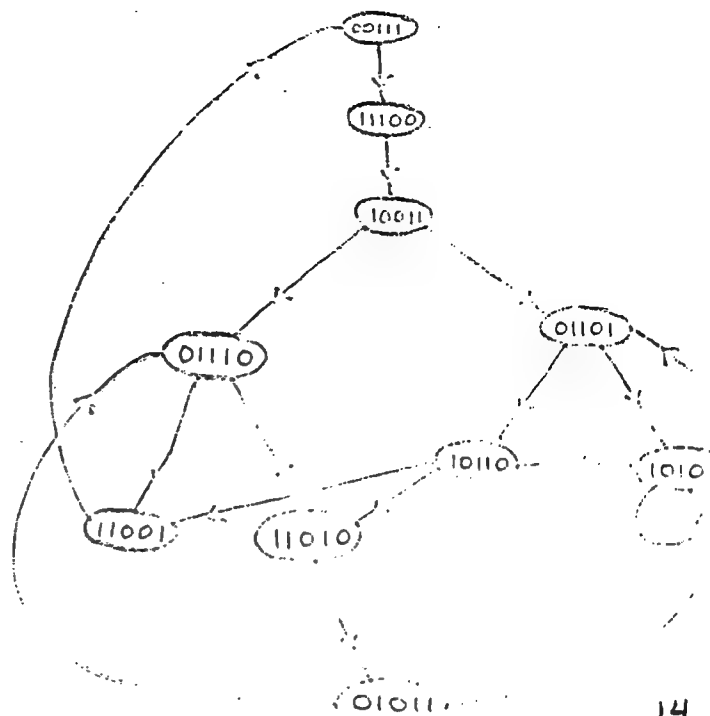
Hence, for n odd, both graphs are connected and the graphs are Eulerian and Hamiltonian, respectively.

n even

$$
\begin{array}{cccccccccc}
\underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\underline{0}\ \underline{0}\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\
\underline{0}\ \underline{0}\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\
\underline{0}\ \underline{0}\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\
\underline{0}\ \underline{0}\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\
\underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\end{array}
$$

16

Cycle thru from component to component

Count the components:

Polya's Thm (Burnside) Let G be a group of permutations acting on A, and let S be the equivalence relation on A induced by G

$$\#E.C. = \frac{1}{|G|}\sum_{\pi \in G} Inv(\pi)$$

$G = Z_{\frac{n}{2}}$.

A = "family" of k element subsets
   with (10 = 01),

Equivalence Classes = Components of G.

17

Example

3-element subsets of an 8-element set.

$$a_1\, a_2 \quad a_3\, a_4 \quad a_5\, a_6 \quad a_7\, a_8$$

One pair = 11   and one pair = 01

or

Three pairs = 01

$$|A| = (12 + 4) = 16.$$

$$\# \text{Components} = \frac{1}{4}(16 + 0 + 0 + 0) = \underline{4}$$

18

A final message from our sponsor . . .

" We need the k - element subsets
   of an n - element set !! "

When n is odd, find the cycle, and generate
the sets...

When n is even, find the k element subsets
of an n+1 element set,
throw out the subsets with n+1.

19

What's really the story ??

Good's result is an existence Theorem.

How do you find the Eulerian Circuit ??

20

For DeBruijn Sequences ...

Fredrickson has given a "linear" (in n)

algorithm to generate the sequence.

For these generalized DeBruijn Sequences,

worst case o(n^k)

average case o(logn)

Best Possible ??

21

Hochberg, Hurlbert and Isaak have also

discovered the idea of multiple shifting.

Carla Savage uses this idea to generate

"new" Grey Codes...

The idea "works" to generate the n!

permutations of an n set.

22

"Matchings in the Partition Lattice"

E. Rodney Canfield, University of Georgia
Department of Computer Science, Athens, GA

Matchings in the Partition Lattice

Rod Canfield

Univ. of Georgia

1.

$$[n] = \{1, 2, \dots, n\}$$

partition $\quad \pi = \{B\} \qquad \Pi_n$

$$B_i \cap B_j = \emptyset \qquad \cup B_j = [n]$$

$$\pi_1 \leq \pi_2$$

$$\left\{ \{1,5\}, \{2\}, \{3,8\}, \{4,7\}, \{6\} \right\} \leq \left\{ \{1,4,5,7\}, \{2\}, \{3,6,8\} \right\}$$

$$E \subseteq \Pi_{n,k} \times \Pi_{n,k+1}$$

$$d_E(x) = \sum_{B \in x} (2^{|B|-1} - 1) \qquad d_E(y) = \binom{k+1}{2}$$

ϕ — 1 — 1,2 — 1,2,3

2 — 2,3

3 — 1,3
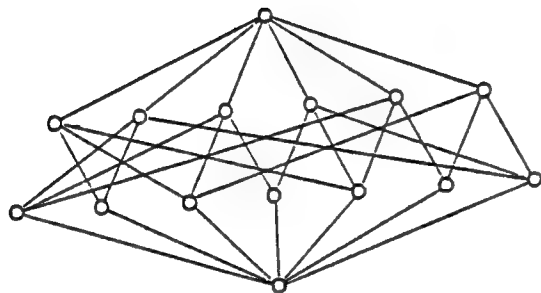
ϕ — 1 — 1,2 — 1,2,3 — 1,2,3,4

2 — 2,3 — 2,3,4

3 — 1,3 — 1,3,4

4 — 1,4 — 1,2,4

2,4

3,4

Chain decompositions in the Boolean lattice B_n, $n = 3, 4$, based on the recursion

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$
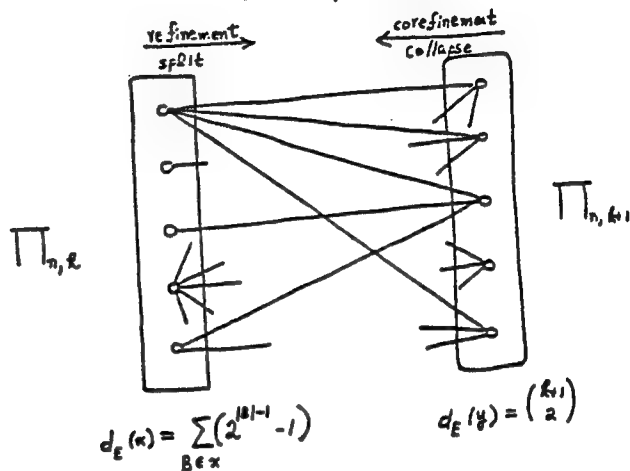
$$|\Pi_{n,k}| = S(n,k)$$

$$S(n+1, k) = \underline{k\, S(n,k)} + S(n, k-1)$$

$$\left\{ \{1\}, \{2,3\} \right\}$$

$$\left\{ \{1,4\}, \{2,3\} \right\} \quad \left\{ \{1\}, \{2,3,4\} \right\} \quad \left\{ \{1\}, \{2,3\} \right. \cdots$$

log concave : $S(n,k)^2 \geq S(n, k-1)\, S(n, k+1)$

$\therefore$ unimodal

```
                    1
                  1   1
                1   3   1
              1   7   6   1
            1  15  25  10   1
          1  31  90  65  15   1
        1  63  301  350  140  21   1
      1  127  966  1701  1050  266  28   1
    1  255  3025  7770  6951  2646  462  36   1
  1  511  9330  34105  42525  22827  5860  750  45   1
```

Stirling numbers of the second kind

③
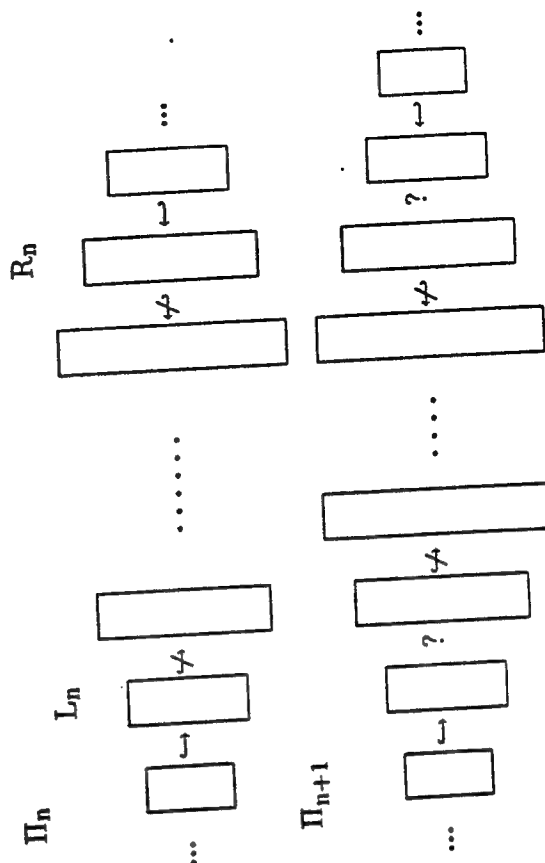
Construct chains using partition recursion. <u>Claim</u> $\exists$ sharp threshold L_n s.t. $\Pi_{\jmath,k} \hookrightarrow \Pi_{n,k+1} \iff k < L_n$.

First, recall Phillip Hall Criteria

$$X \hookrightarrow Y \iff \forall S, \ |S| \le d_E(S)$$

$$E \subseteq X \times Y$$

<u>LMPH</u> $\quad d_E(x) \ge d_E(y) \Rightarrow X \hookrightarrow Y.$

7

Proof that $L_{n+1} = L_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
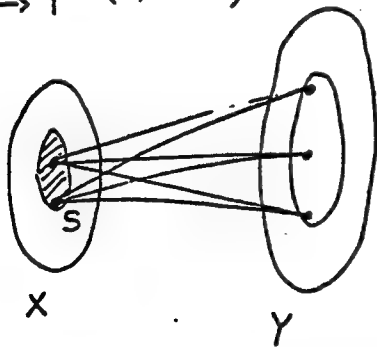
$$S \subseteq \Pi_{n+1,k} \qquad k < L_n$$

$$|S| = \sum_{j=0}^{k} |S_j|,$$

$$S_0 \subseteq \Pi_{n,k-1}$$

$$S_j \subseteq \Pi_{n,k}$$

$$d_E(S) \ge \sum_{j=0}^{k} d_E(S_j)$$

$$\ge \sum_{j=0}^{k} |S_j|$$

$$= |S|.$$

That's <u>half</u> of the proof.

8

Claim $\exists\, R_n$ s.t. $\Pi_{n,k} \hookrightarrow \Pi_{n,k-1} \Leftrightarrow k > R_n$.

Proof Let $A \subseteq \Pi_{n,k} \cup \Pi_{n,k-1}$ be an antichain, w/ $k > R_n + 1$. Want to show $|A| \le S(n, k-1)$.

For each B, $\phi \subseteq B \subseteq [n]$, let A_B be those partitions π of $[n] - B$ s.t.

$$\pi \cup \{B \cup \{n+1\}\} \in A.$$

$$A_B \subseteq \Pi_{n-|B|, k-1} \cup \Pi_{n-|B|, k-2}$$

and A_B is an antichain.

Hence, $|A_B| \le S(n, k-2)$, and

$$||A| = \sum_B |A_B|$$

$$\le S(n, k-2) + \sum_{\phi \ne B \subseteq [n]} S(n-|B|, k-2)$$

$$= S(n, k-2) + (k-1)\, S(n, k-1)$$

$$= S(n+1, k-1) \qquad \blacksquare$$

Theorem There exist monotone increasing sequences L_n and R_n such that

$$\Pi_{n,k} \hookrightarrow \Pi_{n,k-1} \Leftrightarrow k < L_n$$

$$\Pi_{n,k} \hookrightarrow \Pi_{n,k-1} \Leftrightarrow k > R_n.$$

As n increases by 1, each of L_n, R_n grows by at most 1.

Next, want to give bounds for L_n and R_n. What can we find using only LMPH?

Fact 1 For $k \le n \log 2 / \log n$ and $n \ge 5$, $\Pi_{n,k} \hookrightarrow \Pi_{n,k+1}$.

Hence, $L_n > n \log 2 / \log n$, $n \ge 5$.

Proof $d_E(x) = \sum_{B \in x} \left(2^{|B|-1} - 1\right)$

$$\ge k\left(2^{\frac{n}{k}-1} - 1\right)$$

$$\ge k\left(\tfrac{n}{2} - 1\right)$$

$$\ge \tfrac{1}{2} k\,(n \log 2 / \log n + 1), \quad n \ge 6$$

$$\ge \binom{k+1}{2}$$

$$= d_E(y)$$

Fact 2 Fix $\delta > 0$. If n is suff'ly large and $k \ge (1+\delta) n \log 4 / \log n$, then $\Pi_{n,k} \hookrightarrow \Pi_{n,k-1}$. Hence,

$$R_n < (1+\delta) n \log 4 / \log n.$$

Proof. Curiously, we throw away edges. Take $\delta \le \tfrac{1}{2}$; define

$$b = \left\lfloor \left(1 - \tfrac{\delta}{2}\right) \tfrac{\log n}{\log 4} \right\rfloor.$$

Let $E = (y, x) \in \Pi_{n, k-1} \times \Pi_{n, k}$ s.t. x is obtained from y by splitting a block of size $\le 2b$.

Say $x \in \Pi_{n,k}$; since

$$k\,(b+1) \ge \left(1 + \tfrac{\delta}{4}\right) n, \quad \text{at least}$$

$$\left(1 - \left(1+\tfrac{\delta}{4}\right)^{-1}\right) k \ge \tfrac{2\delta}{9} k \quad \text{blocks of}$$

Left column (page 13)

x · are of size $\leq t$. If $\delta k \geq \sqrt{5}$,

$$d_E(x) \geq \tfrac{1}{45}\,\delta^2 k^2.$$

On the other hand,

$$d_E(y) \leq k\,2^{2t}$$
$$\leq k\,n^{1-\frac{\varepsilon}{2}}.$$

Hence, if

$$\frac{\delta^2}{45}(1+\delta)\,\log 4\,/\,\log n \;\geq\; n^{-\delta/2},$$

$$d_E(x) \geq d_E(y). \qquad \blacksquare$$

Right column (page 14)

How far from the truth might these two bounds be?

$$\underline{\text{Thm}}\qquad \frac{L_n}{n/\log n} \to \log 2$$

$$\frac{R_n}{n/\log n} \to \log 4$$

Further research

1. For what n do we have
$$|\{L_n, K_n, R_n\}| \geq 2$$
(Let $n_0 =$ smallest)

2. $\max\{|A| : A \subseteq \Pi_n,\ \text{antichain}\} \sim \;?$

3. other

13

1·

Left column (page 15)

$g,\ h$ polynomials

$$Z = X_1 +\cdots+ X_\ell + Y_1 +\cdots+ Y_{k-\ell} \qquad \text{random var.}$$

$$P\{X=j\} = \frac{c_j n^j}{g(n)} \qquad \mu_g = E(X) \qquad \sigma_g^2 = E(X^2) - E(X)^2$$

If $\quad \ell\mu_g + (k-\ell)\mu_h = n$

then $\quad [z^n]\, g(x)^\ell\, h(x)^{k-\ell} = \dfrac{g(n)^\ell\, h(n)^{k-\ell}}{n^n}\, P\{Z=n\}$

$$\approx \frac{g(n)^\ell\, h(n)^{k-\ell}}{n^n}\,\frac{1}{\sigma\sqrt{2\pi}}$$

by $\underline{\text{CLT}}$, with $\quad \sigma^2 = \ell\,\sigma_g^2 + (k-\ell)\,\dfrac{\sigma^2}{h}.$

$A \subseteq \Pi_{n,k}$: ℓ blocks of size $1..m$, and $(k-\ell)$ of size $m+1..2m$

$$|A| = \frac{n!}{\ell!\,(k-\ell)!}\,[x^n]\, g(x)^\ell\, h(x)^{k-\ell}$$

w/ $\quad g(x) = \sum_1^m x^i/i!$

15

Right column (page 16)

refine $(A) \subseteq C_1 \cup C_2$
$$\quad \vdash\ \ell+2,\ k-\ell-1$$
$$\quad \vdash\ \ell+1,\ t-1$$

Given sequence (n, k) w/ $n \to \infty$,

$$k \sim \beta\, n/\log n, \qquad \beta > \log 2:$$

exhibit $\quad n,\ \ell,\ m$ s.t.

(1) estimation procedure works

(2) same r usable for A, C_1, C_2

(3) all three σ's are $\sim$

Find

$$\frac{|C_1| + |C_2|}{|A|} \;\sim\; \frac{g(n)}{\ell+1} + \frac{g(n)^2}{(\ell+2)(\ell+1)}\,\frac{(k-\ell)}{h(n)}$$

$$\to 0.$$

16

Say $k \le \cdot (1-\delta)\, n \log 4 / \log n$

Let $m = \lfloor (1+\delta) \log n / \log 4 \rfloor$

$S = \{ \pi \in \Pi_{n,k} :$

$\qquad \text{blocksize}(\pi) \subseteq \{n, 2m, 3m, 3m+1\ldots\}$

$\qquad \ge n^{1-\delta/2}$ blocks of size $2m$

$\qquad \ge n^{1-\delta/2}$ blocks of size $\ge 3m \}$

$\frac{1}{2}\binom{2m}{m} n^{1-\delta/2}$ and $\binom{7m}{m}$ are

both $\Omega(n^2)$. Hence,
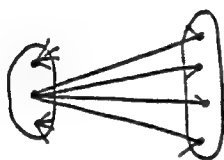
corefine(S) $\qquad S \subseteq \Pi_{n,k}$,

and $\Pi_{n,k} \not\to \Pi_{n,k-1}$.

57

Knowledge about Π_n

1928 Sperner : The subset lattice is Sperner

1967 Rota : Is Π_n Sperner ?

1967 Harper : Asymptotic normality, $K_n \sim n/\log n$

1968 Lieb : Log concavity

1969 Mullin : K_n is the critical part

1969 Graham, Harper : $\Pi_{13,5} \to \Pi_{13,6}$ via Phillip Hall quotient reduction

1971 Dilworth, Greene : geometric $\neq$ Sperner ($> 60{,}000$ elements)

1971 Kleitman, Edelberg, Lubell : Antichains are symmetric

1974 Harper : Π_n is LYM for $n \le 19$

1974 Spencer : Π_n is not LYM for $n \ge 20$

1977 Pudlak, Tuma : Every lattice can be embedded in Π_n for some n

1977 Canfield : Π_n is not Sperner, $n_0 \le$ (?) 6.5×10^{24}

1979 Shearer : Π_n is not Sperner, $n_0 \le 3.7 \times 10^6$

1980 Canfield : $K_n = \ldots$

1980 Shearer : Maximum antichains intersect $\Pi_{n,k}$ for $k \ge (1-\delta)n \log 4 / \log n$

1984 Jiang, Kleitman : Π_n is not Sperner, $n_0 \le 3.4 \times 10^4$

1985 Harper : $e_n \ge$ (?) $(1 - 3\sqrt{5}/5)^{-1/2} S(n, K_n)$

1990 Kung : $\Pi_{n,k} \to \Pi_{n,k-1}$ for $k > n/2$

1991 Canfield : L_n and R_n

"Algorithms for Small Graphs"

Ronald C. Read, University of Waterloo
Department of Combinatorics and Optimization, Ontario, Canada

Algorithms

for

small graphs

Ron Read

UNIVERSITY OF WATERLOO

ATLAS OF GRAPHS

Graphs ≤ 7	1252	Digraphs ≤ 4	
Trees ≤ 12	987	Tournaments ≤ 7	5
Line graphs ≤ 8	357		
Identity trees ≤ 14	1261	Eulerian digraphs ≤ 5	
Homeo. irred. trees ≤ 16	568	2-reg. digraphs ≤ 7	
Unicyclic graphs ≤ 8	253	Acyclic digraphs ≤ 5	
2-conn. planar ≤ 7	639	Self-comp. digraphs ≤ 5	
3-conn. planar ≤ 8	301	Signed graphs ≤ 5	
Cubic graphs ≤ 14	621		
Quartic " ≤ 11	350	SPECIAL GRAPHS	
5-reg. " ≤ 10	64	Platonic	
Eulerian graphs ≤ 8	236	Archimedean	
Self-comp. " ≤ 9	49	Snarks	
Outerplanar " ≤ 9	372	Symmetric graphs	
		Counter-examples	
		etc.	

2

Graphs with 7 vertices (continued)

3

Trees with 12 vertices (continued)

Homeomorphically irreducible trees (continued)
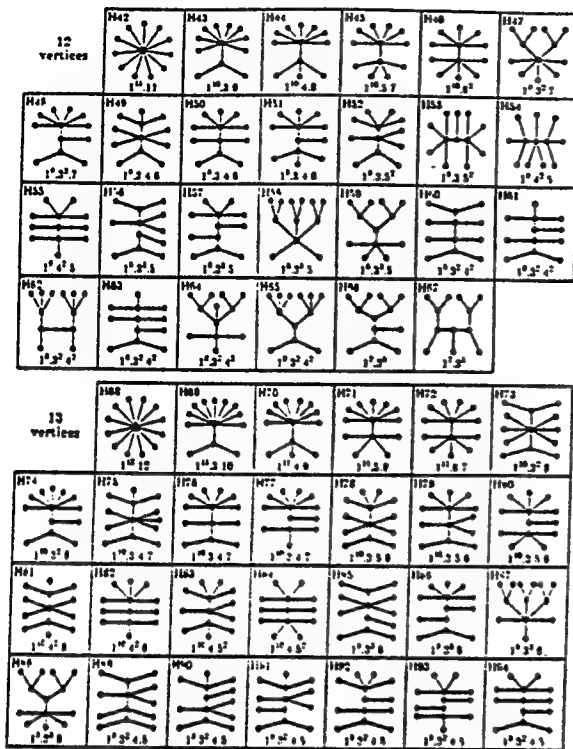
4-regular graphs with 11 vertices (continued)
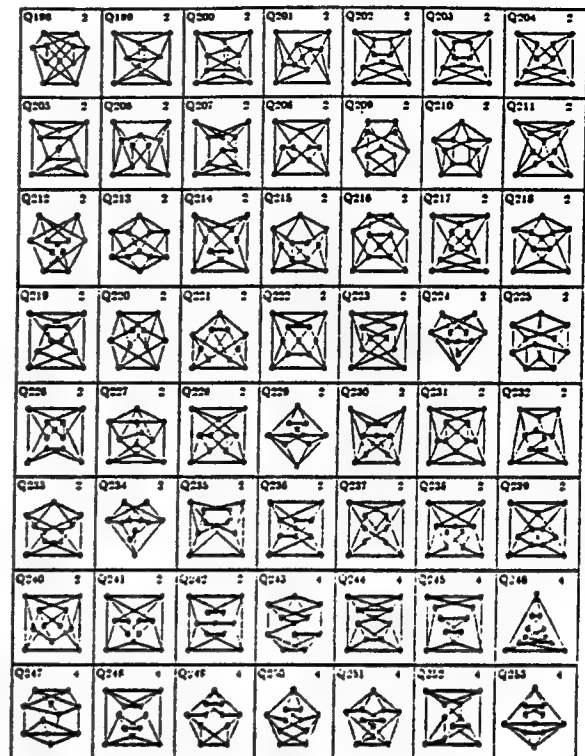
Self-complementary digraphs (continued)
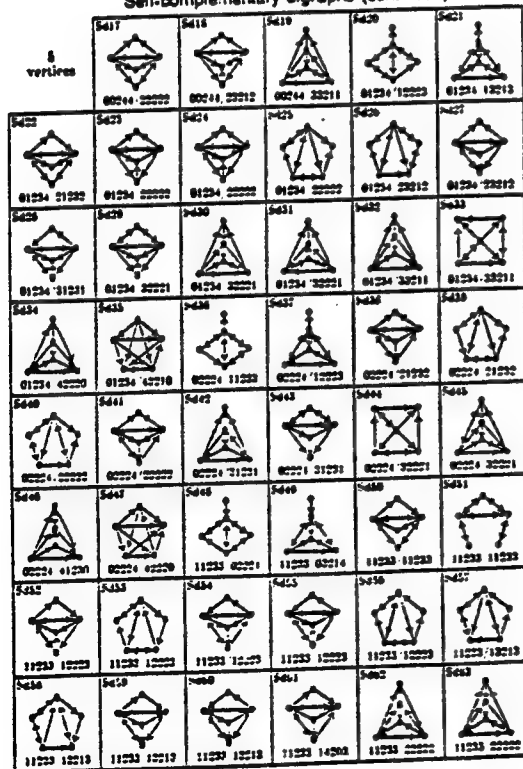
2-regular digraphs

Special graphs. 6

Falkner's graph

Franklin's graph

The Horwood graph

McGee's graph

Parameters and Properties.

p = no. of vertices q = no. of edges.

k = no. of components

Degree sequence

g = girth   (girth number)

Circumference

Diameter

κ = vertex connectivity

λ = edge connectivity

Order of automorphism group

Eulerian ?

Hamiltonian ?

Planar ?

Bipartite ?

Tree ?

Chromatic number χ

Edge chromatic number χ'

Chromatic polynomial

Characteristic polynomial

Spectrum

Characteristic polynomial

Adjacency matrix A

$$\phi(x) = |xI - A|.$$

Newton's method.

Let q_j = trace A^j

and $\phi(x) = \sum_{k=0}^{n} p_k x^k$.

Then $k p_k = - \sum_{j=0}^{k-1} p_j q_{k-j}$

Example:
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
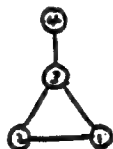
$q_1 = 0, \quad q_2 = 8, \quad q_3 = 6, \quad q_4 = 28$

$p_0 = 1, \quad p_1 = 0, \quad p_2 = -4$

$p_3 = -2, \quad p_4 = 1$

$\phi(x) = x^4 - 4x^2 - 2x + 1$

SPECTRUM

= set of eigenvalues of the adjacency matrix A.

= set of roots of $\phi(x) = 0$

Roots lie between $-(n-1)$ and $n-1$.
All roots are real.

Find integer roots first. Then use Newton's formula, starting with the largest. As each root is found, take out the linear factor.

Newton's formula

If α is an approximate root, then

$$\alpha' = \alpha - \frac{\phi(\alpha)}{\phi'(\alpha)}$$

is a better one (under appropriate conditions)
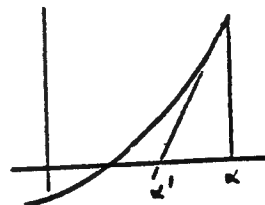
Chromatic Polynomial

$P_G(\lambda)$ = number of ways of coloring G with λ colors available.

Chromatic reduction:

$$P_G(\lambda) = P_{G'}(\lambda) - P_{G''}(\lambda)$$
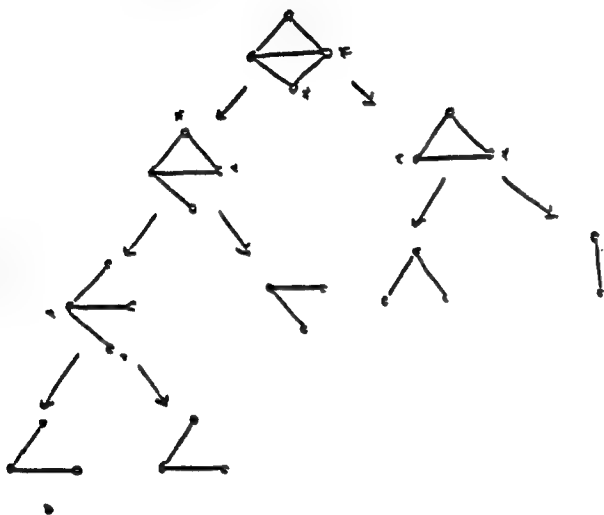
If $k=1$ and $p-q=1$, G is a tree.

$\chi(G)$ is the least integer λ such that $P_G(\lambda) > 0$.

Edge-chromatic number $\chi'(G)$

VIZING's THEOREM: $\chi'(G) = \Delta$ or $\Delta+1$.

METHOD Try to color the edges of G in Δ colors. If successful, $\chi'(G) = \Delta$. Otherwise $\chi'(G) = \Delta+1$.

Start by allocating colors to the edges at a vertex of degree Δ.
Thereafter: For every edge not yet colored there will be a set of allowable colors. Choose an edge, give it an allowable color.

Continue, backtracking when an edge cannot be colored.

Stop when only 2 edges are left.
This works even if G is not connected.

Properties.

If $P_G(2) > 0$, G is bipartite.

The lowest power of λ with non-zero coefficient is k — the number of components.

If g is the girth of G, the coefficients (in absolute magnitude) are

$$1, q, \binom{q}{2}, \binom{q}{3}, \ldots, \binom{q}{g-2}$$

and the next coefficient is less than $\binom{q}{g-1}$ by the number of cycles of length g.

DIAMETER

Diameter = greatest distance between two vertices.

$$= \max_{u,v} \left\{ \min_{\substack{\text{all } u-v \\ \text{paths}}} (\text{pathlength}) \right\}$$

Method Find all (u,v)-distances using an algorithm based on a theorem of Warshall.

For $k = 1$ to p
do: for $i = 1$ to p and $\neq k$
     for $j = 1$ to p and $\neq k$
       $d(i,j) \leftarrow \min(d(i,j), d(i,k)+d(k,j))$

(All $d(i,i)$ initially $= \infty$)

CIRCUMFERENCE

Circumference = length of a longest cycle

NP-complete (includes the Hamilton cycle problem)

Method: Start at some vertex x. Try to construct a Hamilton cycle, but keep track of the longest cycle found.

If this is p.   circumference is p
— graph is Hamiltonian

If it is $p-1$ circumference is $p-1$
— graph is not Hamiltonian

Otherwise. Repeat with the graph $G - x$.

Short cuts

Eliminate vertices of degree < 2.
A vertex of degree 2, if included, implies inclusion of its incident edges.

Planarity.

Hopcroft & Tarjan 1974   (linear).
Too elaborate!

Preliminaries

Theorem If $q > 3p - 6$ G is non planar.

If
$q - p = -1$ G is a tree
$q - p = 0$ G is unicyclic
$q - p = 1$ G has two indept. cycles
$q - p = 2$ G has 3 " "

— all these must be planar.

The case of $K_{3,3}$ ($p=6$, $q=9$) shows that $q - p = 3$ does not imply planarity.

If $p \le 4$, or $p = 5$ and $G \ne K_5$ then G is planar.

The $q > 3p - 6$ criterion is obscured by the presence of vertices of degree 2.

Method Delete all vertices of degree 1. "Smooth out" vertices of degree 2.
(continue if possible)

If G is disconnected, keep components with > 5 vertices. (For $p \le 7$ there will be only one at most)
(Assume G connected).

Perform the preliminary tests.

If planarity/nonplanarity is still not determined, perform a planarity test.

Which one?

The Fisher/Wing algorithm.

Special case when G is Hamiltonian
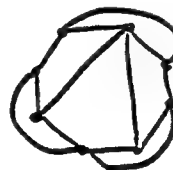
Easy to compute whether the chords are compatible.

What if the graph is not Hamiltonian?

(ASK)

Connectivity (K and λ)

Information from the chromatic polynomial.

1. If the coefficient of λ is zero.
   — G is not connected.

2. If $P_G(\lambda)$ is divisible by $(\lambda-1)^2$

   then G has a cut vertex

Otherwise G is <u>at least</u> 2-connected, but there seems to be no short cut to finding the exact value of K.

———

The value of λ can be found by using the max-flow-min-cut theorem; but no obvious short cuts.

Classify vertices by convenient criteria.
e.g. number of triangles a vertex is on.
(cube of adjacency matrix).
Number of 2-paths between vertices
(square of adjacency matrix).

Run through permutations of vertices which permute sets of equivalent vertices.

Number of automorphisms

For graphs with $p \leq 10$ this number was computed and recorded when the graph was generated.

For regular graphs

Example. 4-regular (quartic) graphs, $p = 11$.

These were generated by extracting from the 10-vertex catalog those graphs with degree sequence

4 4 4 4 4 4 3 3 3 3

and joining the four vertices of degree 3 to a new vertex of degree 4.

This gives each required graph at least once.

Now eliminate duplicates

How?

"Characterization of Generalized Bicritical Graphs"

Nathaniel Dean, Bellcore
Morristown, NJ

Characterization of Generalized Bicritical Graphs

Nate Dean

Bellcore
Morristown, NJ 07960

1
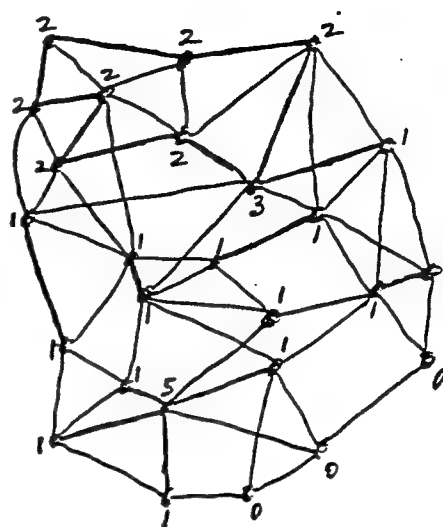
Definitions

f - Factor

2

Definitions
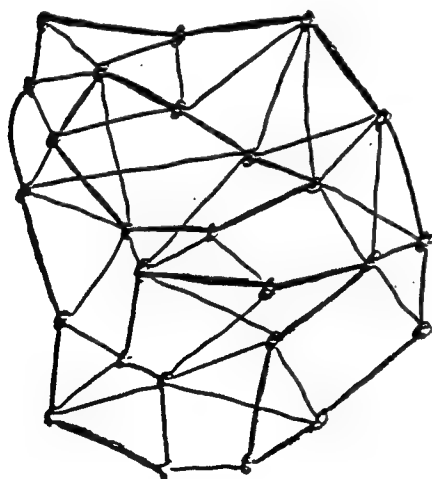
2-factor

3

Definitions
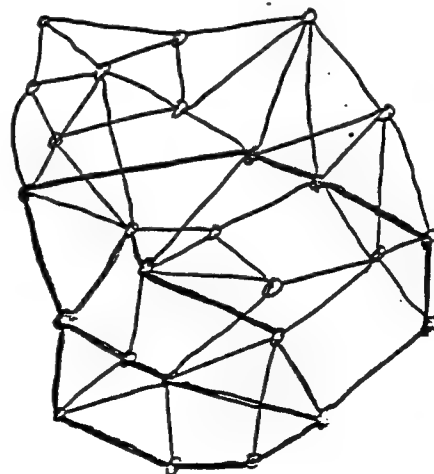
4-connectivity

4

Applications

- Lower bound for TSP

- Degree sequences
  - Construct graphs
  - Prove theorems

- Easy proof of Tutte's "4 conn., planar $\Rightarrow$ Ham." theorem?

5

1-Factor Theorems

Menger - 1927
König & Egervary - 1931
Hall - 1935

A bipartite graph $G = (V_1, V_2 ; E)$ with $|V_1| = |V_2|$ has a 1-factor $\Leftrightarrow |N(S)| \geq |S| \quad \forall S \subseteq V_1$.

Tutte - 1947

G has a 1-factor $\Leftrightarrow c_0(G-S) \leq |S| \quad \forall S \subseteq V(G)$.

6

Bicritical Graphs
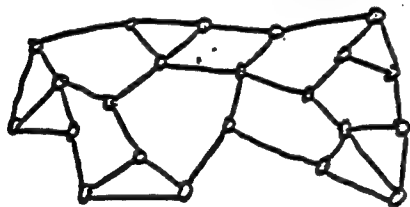
Even Halin Graphs
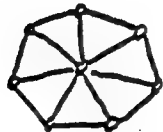
Wheels

7

Special Graphs

- **Elementary Graph:**
  Has a p.m., and the edges contained in a p.m. form a connected subgraph.

- **Bicritical Graph:**
  $G - u - v$ has a p.m. for all vertices u, v.

8

Slide 9
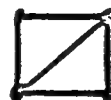
What is a bicritical graph?

- Lovász: G is bicritical iff
  $\underline{S \subseteq V(G)}$ and $|S| \geq 2 \Rightarrow c_0(G-S) \leq |S|-2$.

- Bicritical $\Rightarrow$ 2-connected with $\delta \geq 3$.

- G is bicritical iff
  each $\underline{\text{split graph } G_i}$ of G is bicritical.

$$G \longrightarrow G_1 \qquad G_2$$

9

Slide 10
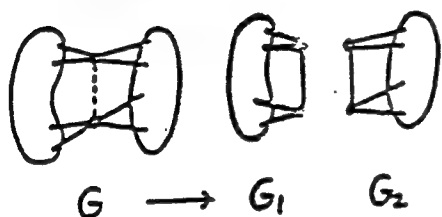
Examples - Lovász and Plummer

- Every $\underline{\text{even Halin graph}}$ is bicritical.

- G is $\underline{\text{connected, even, vertex-transitive}}$
  $\Rightarrow G$ is elementary bipartite or bicritical.

- G is $\underline{\text{cyclically } (k+1)\text{-edge connected, even,}}$
  $\underline{k\text{-regular}}$
  $\Rightarrow G$ is elementary bipartite or bicritical.

10

Slide 11

- Corollary (Nishizeki-1978)
  Every even, 4-connected, projective-planar
  graph $\underline{\text{has a perfect matching}}$.

- $\exists$ even, 4-connected, $\underline{\text{toroidal}}$ graphs which
  are $\underline{\text{not bicritical}}$. Example: $C_{2m} \times C_{2n}$.

11

Slide 12
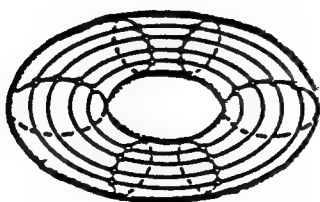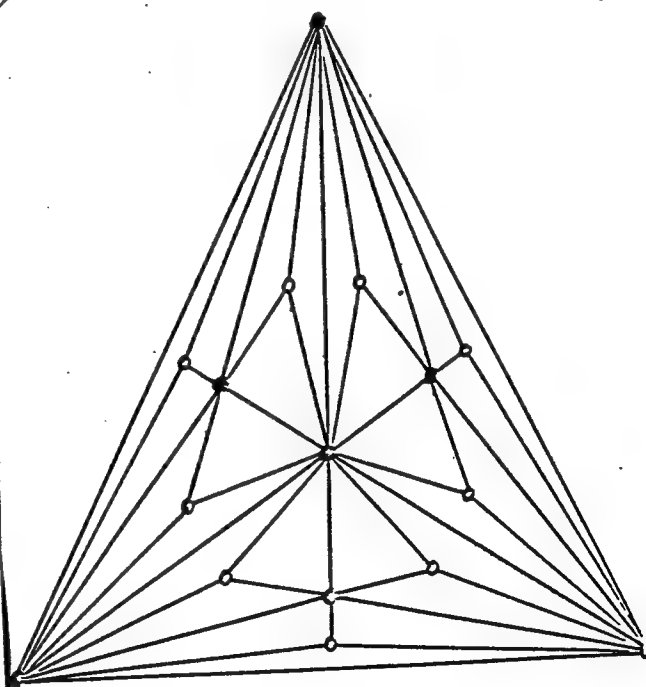
$$c_0(G-S) > |S| \Rightarrow \text{ no p.m.}$$

12
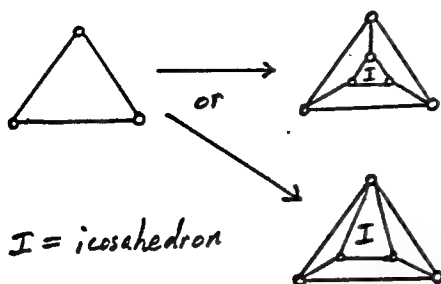
I = icosahedron

3-connected
planar
min degree = 5
$|V(G)|$ even
$\Big\}$ $\not\Rightarrow$ p.m.
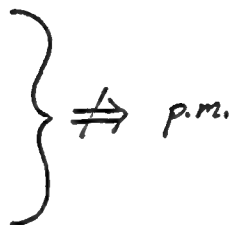
Characterization

Let G be 4-connected and embeddable in the torus or the Klein bottle. Then G is bicritical iff G is even and

(1) G is projective-planar or

(2) G has no set $S \subseteq V(G) \ni$

$G-S$ has no even components and contracting each component of $G-S$ yields a graph G' with $|S| = |V(G')-S|$ and with $G' - E[G[S]]$ 4-regular.

$\leftarrow$ no even components

S
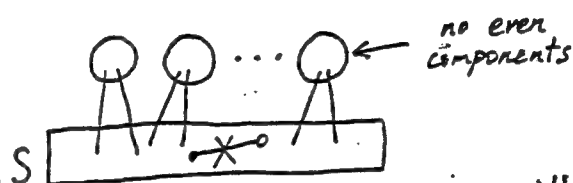
Matching Extendability

- n-extendable: $p \geq 2n+2$, G has a p.m., and every matching of size n is contained in a p.m.

- Matching extendability $\mu(\Sigma)$ of a surface Σ: smallest integer $n \ni$ no graph embeddable in Σ is n-extendable.

- Plummer: $\mu(\Sigma) = ?$ for orientable Σ.

- Answer: $\Sigma \neq$ sphere $\Rightarrow$

$$\mu(\Sigma) = 2 + \left\lfloor \sqrt{4 - 2\chi} \right\rfloor.$$

Telecommunications
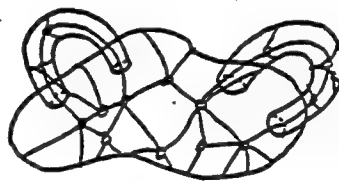
- Irregularly shaped surface
- Nodes communicate along ≤ 1 edge
- Can matching be extended to one where every node communicates?
- To what extent does the surface obstruct the extension?

Sufficient Conditions for Hamiltonian Cycle

Whitney (1931)
   4-connected plane triangulation

Tutte (1956)
   4-connected, planar

Duke (1972)
   1) 2-conn., toroidal, $\delta \geq 6$
   2) 2-conn., toroidal, $\delta \geq 4$, triangle-free

Thomas & Yu (1991)
   4-conn., projective planar

17

Conjectures

Grunbaum (1970) & Nash-Williams (1973)
   4-conn., toroidal $\Rightarrow$ Ham.

Molluzzo (1979) — Negami & Ota
   6-conn., toroidal $\Rightarrow$ Ham.-conn.

Grunbaum (1970) — Thomas & Yu
   4-conn., projective planar $\Rightarrow$ Ham.

Dean (1990)
   4-conn., proj. planar $\Rightarrow$ Ham.-conn.
   5-conn., toroidal $\Rightarrow$ Ham.-conn. ?

18

Tutte's f-Factor Theorem

$$\delta(f,S,T) \triangleq$$
$$\sum_{x \in S} f(x) + \sum_{x \in T} d_G(x) - e(S,T)$$
$$- \sum_{x \in T} f(x) - h(f,S,T)$$

where

$h(f,S,T) \triangleq$ number of components C
of $G - S - T \ni$

$$\sum_{x \in V(C)} f(x) + e(V(C),T) \text{ is odd.}$$

G has an f-factor iff $\delta(f,S,T) \geq 0$
$\forall$ disjoint $S, T \subseteq V(G)$.

19

f-Bicritical Graphs

Let $f: V(G) \to \mathbb{Z}$.
G is f-bicritical if $\forall u,v \in V(G)$,
$G - uv$ has an $f_{u,v}$-factor where
$f_{u,v}(x) = f(x)-1$ if $x \in \{u,v\}$ and
$f_{u,v}(x) = f(x)$ otherwise.

Note:
• G is f-bicritical $\Rightarrow$ every edge
lies in an f-factor.
• G is 1-bicritical $\Leftrightarrow$ G is bicritical
(i.e., $\forall u,v \in V(G)$, $G-u-v$ has a p.m.)

Theorem. $\delta(f,S,T) \geq 2$, $\forall S,T \subseteq V(G)$
with $S \cap T = \emptyset$ and $S \cup T \neq \emptyset \Rightarrow$ f-bicritical.

20

CONJECTURE

Grunbaum - 1970

Nash-Williams -1973

Every 4-connected, toroidal graph is hamiltonian.
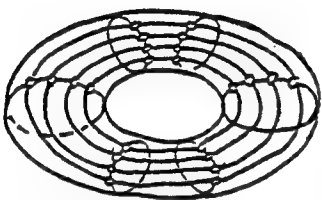
21

Surfaces

- Sphere
- Torus, ...
- Projective plane, Klein bottle , ...

(1) G is bipartite and embeddable in the P.P. or the K.B. or torus $\Rightarrow |E(G)| \leq 2|V(G)|$.

(2) $G \to \Sigma$ are closed under minors.

22

Graph Minors Family $\mathcal{Z}_k$

(1) Closed under minors

(2) Every bipartite member B satisfies $|E(B)| \leq 2|V(B)| - k$.

Examples

- $\mathcal{Z}_0 \supseteq \{ \text{toroidal graphs} \}$
- $\mathcal{Z}_0 \supseteq \{ \text{Klein bottle graphs} \}$
- $\mathcal{Z}_2 \supseteq \{ \text{projective planar graphs} \}$
- $\mathcal{Z}_4 \supseteq \{ \text{planar graphs} \}$

$$\mathcal{Z}_0 \supseteq \mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \mathcal{Z}_3 \supseteq \mathcal{Z}_4$$

23

2-Factors in 4-Conn. G

$G \in \mathcal{Z}_0 \Rightarrow G$ has a 2-factor.

$G \in \mathcal{Z}_1 \Rightarrow$

(1) G is 2-bicritical.

(2) $G - u$ has a 2-factor, $\forall u \in V(G)$.

(3) $G - u - v$ has a 2-factor, $\forall u, v \in V(G)$.

24

Sample Theorem
(with K. Ota)

G is 4-connected and embeddable in the torus or the Klein bottle
$\Rightarrow$ G has a 2-factor.

Best possible: $K_{4,n}$, $n \geq 5$
- 4-connected
- not embeddable in torus
- not embeddable in Klein bottle
- no 2-factor

-25

Tutte's 2-Factor Theorem
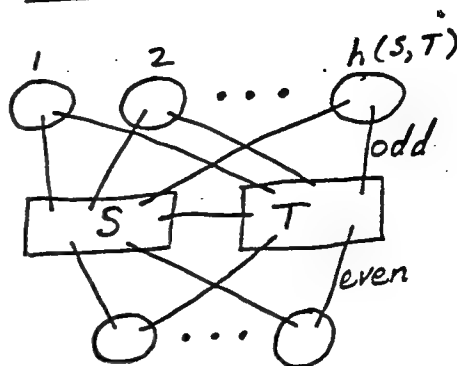
$\delta(S,T) \triangleq$
$$2|S| + \sum_{x \in T} d_G(x) - e(S,T) - 2|T| - h(S,T)$$

Theorem.
  G has a 2-factor $\Leftrightarrow$
  $\delta(S,T) \geq 0$ $\forall$ disjoint $S, T \subseteq V(G)$.

Note: $\delta(\emptyset, \emptyset) = 0$.

26.

Proof Strategy

- Reduce G to the essentials
- Case $|S \cup T| \leq 3$
- Case $|S \cup T| \geq 4$
- Collect info on odd components
  – definitions + claims

- Substitute into formula for $\delta(S,T)$ to show ≥ 0

27

Reductions
$G \to G_0$
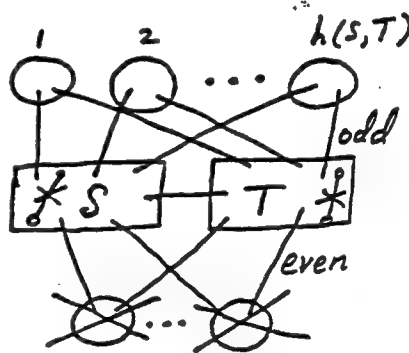
Contract each odd component.

28

Case $|S \cup T| \leq 3$

$G_0:$ (example)

S T
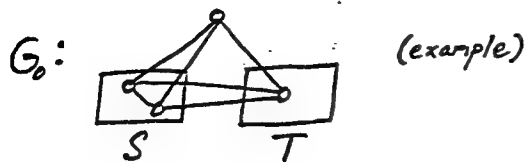
- $h(S,T) \leq 1$ since G is 4-conn.

- $\delta(S,T) \geq 2|S| + 4|T| - e(S,T) - 2|T| - 1$
$$= 2|S \cup T| - e(S,T) - 1$$
$$\geq \begin{cases} 4-2-1, & \text{if } |S \cup T| = 2 \text{ or } 3 \\ 2-0-1, & \text{if } |S \cup T| = 1 \end{cases}$$
$$\geq 1$$

29

Case $|S \cup T| \geq 4$
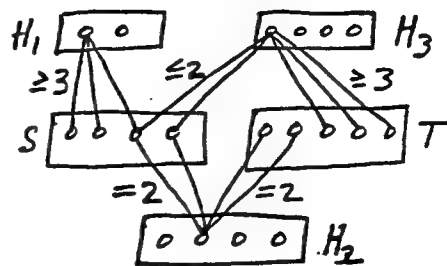
$$h(S,T) = |H_1| + |H_2| + |H_3|$$

Claim.
$$\forall u \in H_2 \ \exists x \in T \ \ni \ e_G(x, C(u)) \geq 2.$$

30

Bipartite Inequality

$= B$
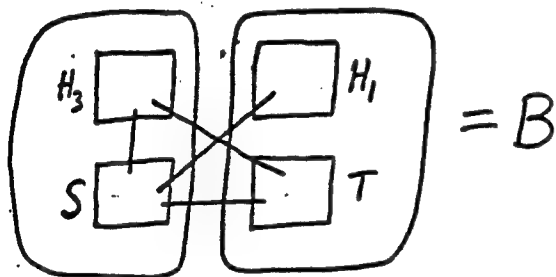
$$|E(B)| \geq e_G(S,T) + 3|H_1| + 3|H_3| + |H_2'|$$
$$|V(B)| = |H_1| + |H_3| + |S| + |T|$$
$$|E(B)| \leq 2|V(B)|$$

$$e_G(S,T) \leq 2|S| + 2|T| - |H_1| - |H_2'| - |H_3|$$

$$\cdots \Rightarrow \delta(S,T) \geq 0.$$

31

Connectivity for 2-Factors

$t(\Sigma) \triangleq$ smallest integer $k \ni$ every k-connected graph embeddable in Σ has a 2-factor.

$$\chi(S_h) = 2 - 2h$$
$$\chi(N_k) = 2 - k$$

Duke (1972)
$$t(\Sigma) \leq 3 + \sqrt{9 - 3\chi}, \text{ if } \Sigma \neq S_0.$$

New Results:
$$t(N_1) = 4 < 5$$
$$t(S_1) = t(N_2) = 4 < 6$$

32

"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

Joseph Straight, SUNY at Fredonia
Department of Mathematics and Computer Science, Fredonia, NY

Extremal Problems Involving Neighborhood Numbers
and Other Graph Parameters

H. Joseph Straight

Department of Mathematics and Computer Science

State University of New York College at Fredonia

Fredonia, New York 14063, USA

ABSTRACT

Given a simple graph $G = (V, E)$, a subset S of V is called a *neighborhood set* provided G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. The minimum cardinality among all minimal neighborhood sets of G is denoted by $n(G)$ and is called the *neighborhood number* of G. It is known, for instance, that $\gamma(G) \leq n(G) \leq \alpha(G)$ for any G without isolated vertices, where $\gamma(G)$ and $\alpha(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $n(p)$ among all connected graphs of order p. Our work so far has lead us to conjecture that

$$n(p) \leq \lfloor 9p/13 \rfloor$$

a result that holds for $2 \leq p \leq 15$. I will report on this work and, as time permits, some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order p having girth at least 5.

Observe that if G is the disjoint union of graphs G_1 and G_2, then

$$\gamma(G) = \gamma(G_1) + \gamma(G_2), \quad n(G) = n(G_1) + n(G_2), \quad \alpha(G) = \alpha(G_1) + \alpha(G_2)$$

We thus assume henceforth that $G = (V, E)$ is connected. Now, if u and v are nonadjacent vertices of G, then

$$\gamma(G + uv) \leq \gamma(G) \quad \text{and} \quad \alpha(G + uv) \geq \alpha(G)$$

However,

$$n(G) - 1 \leq n(G + uv) \leq n(G) + 1$$

For example, consider the graph $G = (\{u, v, w, x, y\}, \{uv, vw, wx, xy, uy\})$. Note that G is a 5-cycle, $n(G + uw) = 2$, $n(G) = 3$, and $n(G - uv) = 2$.

This leads us to consider the following extremal problem: find the maximum neighborhood number $n(p)$ among all connected graphs of order p.

Given a simple graph $G = (V, E)$, the set $N(v) = \{w \in V \mid vw \in E\}$ is called the *neighborhood* of v and $N[v] = N(v) \cup \{v\}$ is its *closed neighborhood*. A subset S of V is called a *neighborhood set* provided G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S. The minimum cardinality among all minimal neighborhood sets of G is denoted by $n(G)$ and is called the *neighborhood number* of G and was introduced by E. Sampathkumar and P.S. Neeralagi [The neighborhood number of a graph, *Indian J. Pure Appl. Math.* 16 (1985), 126–132].

Two related parameters are the (vertex) domination and covering numbers. A subset S of V is a *dominating set* provided every vertex not in S is adjacent to a vertex in it is a *covering set* provided every edge of G has at least one of its incident vertices in Let $\gamma(G)$ denote the minimum cardinality of a dominating set and let $\alpha(G)$ denote the minimum cardinality of a covering set; if G has no isolated vertices, then any covering is also a neighborhood set and any neighborhood set is also a dominating set. Thus, any graph G without isolated vertices,

$$\gamma(G) \leq n(G) \leq \alpha(G)$$

It is natural to wonder whether these parameters are independent, in some sense. is easy to see that $\gamma(G) = 1$ if and only if $n(G) = 1$; the complete graph K_p of order has $\gamma(K_p) = n(K_p) = 1$ and $\alpha(K_p) = p - 1$. It was also observed in the above mention paper that $n(G) = \alpha(G)$ if G is connected and has girth at least 4. However, Jayaram Kwong, and Straight [Neighborhood sets in graphs, *Indian J. Pure Appl. Math.*, to appe gave, for positive integers r, s and t with $2 \leq r \leq s \leq t$, an example of a graph G havi $\gamma(G) = r, n(G) = s$, and $\alpha(G) = t$.

To begin, we find that $n(2) = n(3) = 1$ and $n(4) = 2$. Next, $n(5) = 3$, with the uniqu extremal graph being the 5-cycle. At this point we make the following observation: l G be formed from the disjoint union of G_1 and G_2 by adding a single edge that joins vertex of G_1 with a vertex of G_2. Then $n(G) \geq n(G_1) + n(G_2)$. As a consequence, let be a rational number between 0 and 1 and suppose there exists a graph G_0 of order having $n(G_0) = rp_0$. Then there exist infinitely many values of p for which $n(p) \geq r$ Since we have a graph of order 5 with neighborhood number 3, we tentatively conjectu that $n(p) \leq 3p/5$ (for all $p > 1$).

Two more useful observations. The first is that

$$n(p + 1) \leq n(p) + 1$$

Secondly, suppose G_1 is a connected graph of order p and G_2 is the complete graph order 2. Form the graph G as above. Then G has order $p + 2$ and $n(G) = n(G_1) + 1$. Th we find that

$$n(p) = n(p + 1) \rightarrow n(p + 2) = n(p + 1) + 1$$

Continuing, we find that $n(6) = 3$ and $n(7) = 4$. But now consider the following gra of order 8:

$$J = (\{o, t, u, v, w, x, y, z\}, \{ot, ow, oz, tu, tz, uv, ux, vw, vy, wx, xy, yz\})$$

It is not difficult to show that $n(J) = 5$. It follows that $n(8) = 5$, and we are forced revise our tentative conjecture, namely, we now conjecture that $n(p) \leq 5p/8$.

2

3

Let p be fixed and suppose the value of $n(p)$ is known. As a consequence of the observations made on the preceeding page, if one claims that $n(p+1) = n(p) + 1$, then one must give an example of a graph G of order $p + 1$ having $n(G) = n(p) + 1$. On the other hand, if one claims that $n(p+1) = n(p)$, then one must prove that every connected graph of order $p + 1$ has neighborhood number at most $n(p)$.

We can show that every connected graph of order 9 has neighborhood number at most 5. This, together with $n(8) = 5$, gives us that $n(9) = 5$. Joining two disjoint 5-cycles with an edge yields a connected graph of order 10 having neighborhood number 6; hence, $n(10) = 6$. At this point in our investigation we were able to find a graph of order 11 with neighborhood number 7. This disproves our tentative conjecture that $n(p) \leq 5p/8!$ However, if one writes down a table of values of $n(p)$ for $2 \leq p \leq 11$ the following revised conjecture strongly suggests itself.

Conjecture. For $p \geq 2$,
$$n(p) = n(p+1) \rightarrow n(p+2) = n(p)+1 \text{ and } n(p+3) = n(p+4) = n(p)+2$$

(Unfortunately?) Harris Kwong found a graph H of order 12 having neighborhood number 8, so the conjecture is false for $p = 8$. The graph H can be used to construct a graph of order 13 with neighborhood number 9. Then it can be shown that $n(14) = 9$, also, so that $n(15) = 10$. This is the state of the problem at this point in time. By the way, I still believe that the above conjecture is true for p sufficiently large, say $p \geq 13$.

4

This paper investigates the values of $f(v)$, the maximum number of edges in a graph of order v and girth at least 5. For small values of v we also enumerate the set of extremal graphs. This problem has been mentioned several times by P. Erdős, who conjectured that
$$f(v) = (1/2 + o(1))v^{1/2}v^{3/2}.$$

Given graphs $G_1, G_2, \ldots, G_k$, let $ex(v; G_1, G_2, \ldots, G_k)$ denote the greatest size of a graph of order v which contains no subgraph isomorphic to some $G_i, 1 \leq i \leq k$. One of the main classes of problems in extremal graph theory, known as Turán-type problems, is for given $v, G_1, G_2, \ldots, G_k$ to determine explicitly the function $ex(v; G_1, G_2, \ldots, G_k)$, or to find its asymptotic behavior for large values of v. Thus, the problem we consider in this paper, that of finding the maximum size of a graph of girth at least 5, can be stated as finding the value of $ex(v; C_3, C_4)$.

It is well known that $ex(v; C_4) = \lfloor v^2/4 \rfloor$, and the extremal graph is the complete bipartite graph $K_{\lfloor v/2 \rfloor, \lceil v/2 \rceil}$.

The exact value of $ex(v; C_4)$ is known for all values of v of the form $v = q^2 + q + 1$, where q is a power of 2 [Z. Füredi, Graphs without quadrilaterals, JCT B 34 (1983), 187–190], or a prime power exceeding 13 [Füredi, preprint], and it is equal to $q(q+1)^2/2$. The extremal graphs for these values of v are known. For $1 \leq v \leq 21$, the values of $ex(v; C_4)$ and the corresponding extremal graphs can be found in [C.R.J. Clapham, A. Flockhart, and J. Sheehan, Graphs without four-cycles, JGT 13 (1989), 29–47]. Apparently, the same results were obtained by W. McCuaig in 1984, but were not published. It is well known that $ex(v; C_4) = (1/2 + o(1))v^{3/2}$.

It is important to note that attempts to construct extremal graphs for $ex(v; C_3, C_4)$ by destroying all 4-cycles in the extremal graphs for $ex(v; C_3)$, or by destroying all 3-cycles in the extremal graphs for $ex(v; C_4)$, fail; neither method leads to graphs of order v with $f(v)$ edges.

5

The other results I wish to report on are in the following paper:

Extremal Graphs without Three-Cycles or Four-Cycles

David K. Garnick
Department of Computer Science
Bowdoin College
Brunswick, Maine 04011

Y.H. Harris Kwong
Department of Mathematics and Computer Science
SUNY College at Fredonia
Fredonia, New York 14063

Felix Lazebnik
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716

Abstract

We derive bounds for $f(v)$, the maximum number of edges in a graph on v vertices that contains neither three-cycles nor four-cycles. Also, we give the exact value of $f(v)$ for all v up to 24 and constructive lower bounds for all v up to 200.

5

In this section we present some theoretical results about $f(v)$ and the structure of extremal graphs. Many of them will be used in the subsequent sections. We call a graph G of order v extremal if $g(G) \geq 5$ and $e = e(G) = f(v)$. The following statement gives some simple facts about extremal graphs.

Proposition 2.1. Let G be an extremal graph of order v. Then
(a) G is connected and the diameter of G is at most 3.
(b) If $d(x) = \delta(G) = 1$, then the graph $G - x$ has diameter at most 2.

It turns out that the extremal graphs of diameter 2 are very rare. In fact, it has been shown that the only graphs of order v with no 4-cycles and of diameter 2 are:

The star $K_{1, v-1}$;

Moore graphs: C_5, Petersen graph (the only 3-regular graph of order 10, diameter and girth 5), Hoffman-Singleton graph (the only 7-regular graph of order 50, diameter 2 and girth 5), and a 57-regular graph of order 3250, diameter 2 and girth 5, if it exists;

Polarity graphs.

Remark: The only graphs from the list above which in addition contain no triangles and are regular are the Moore graphs. It is also known that a graph of diameter $k \geq 1$ and girth $2k + 1$ must be regular.

We now derive an upper bound on $f(v)$.

Theorem 2.2. Let G be an extremal graph of order $v \geq 3$ and size e. Then
$$f(v) = e \leq \frac{1}{2}v\sqrt{v-1}$$

Furthermore, equality holds if and only if G is regular and of diameter 2, i.e. G is a Moore graph.

Corollary 2.3. Let G be an extremal graph of order v, size e, and diameter 3. If G is regular, then
$$f(v) = e \leq \frac{1}{2}\sqrt{v^2(v-1) - \frac{5}{2}v}$$

If, in addition, the average degree of G is an integer, then
$$f(v) = e \leq \frac{1}{2}\sqrt{v^2(v-1) - 4v}$$

7

Now we derive a lower bound for $f(v)$. Let q be a prime power, and let $v_q = q^2 + q + 1$, and $e_q = (q+1)v_q$. By B_q we denote the point-line incidence bipartite graph of the projective plane $PGL(2,q)$. More precisely, the partite sets of B_q represent the set of points and the set of lines of $PGL(2,q)$, and the edges of B_q correspond to the pairs of incident points and lines. Then B_q is a $(q+1)$-regular bipartite graph of order $2v_q$ and size e_q. Also $g(B_q) \geq 5$: being bipartite B_q has no 3-cycles, and the existence of a 4-cycle in B_q would mean that in $PGL(2,q)$ there are two distinct lines passing through two distinct points.

Theorem 2.4. Let G be an extremal graph of order v and size e. Let q be the largest prime power such that $2v_q \leq v$. Then

$$f(v) = e \geq e_q + 2(v - 2v_q) = 2v + (q-3)v_q$$

Proof. Consider the graph obtained from B_q by adding to $V(B_q)$ a set of $v - 2v_q$ isolated vertices and connecting each of these to two nonadjacent vertices of B_q, taken from different partite sets (such two vertices always exist, since B_q is not a complete bipartite graph). If the chosen pairs of vertices of B_q are distinct for distinct isolated vertices, then we obtain a graph H of order v, size $e_q + 2(v - 2v_q)$ and girth at least 5. There are $q^2 v_q$ pairs of disjoint vertices of B_q in which two vertices in the pair belong to different partite sets. We claim that $q^2 v_q > v - 2v_q$. Indeed, if it is not the case and $q > 2$, then $v \geq (q^2 + 2)v_q > 8v_q$. According to Bertrand's Postulate, for any integer $n \geq 1$, there is at least one prime number p such that $n < p \leq 2n$. Let $n = q$, and let p be a prime satisfying the inequality $q < p \leq 2q$. Then

$$2v_q < 2v_p \leq 8q^2 + 4q + 2 < 8v_q < v$$

which contradicts our choice of q in the statement of the theorem. For $q = 2$, $2v_2 = 14 \leq v < 2v_3 = 26$ implies that $v - 2v_2 = v - 14 < 12 < 26 = 2^2 v_2$. Therefore, for all prime powers q, $q^2 v_q > v - 2v_q$, and the construction of H described above is possible.

8

Theorem 2.2 states that $f(v) \leq \lfloor v\sqrt{v-1}/2 \rfloor$. For $1 \leq v \leq 10$, we have constructed (C_3, C_4)-free graphs with $\lfloor v\sqrt{v-1}/2 \rfloor$ edges. These graphs are shown in Figure 1. This yields the following theorem.

Theorem 3.1. For $1 \leq v \leq 10$, $f(v) = \lfloor v\sqrt{v-1}/2 \rfloor$.
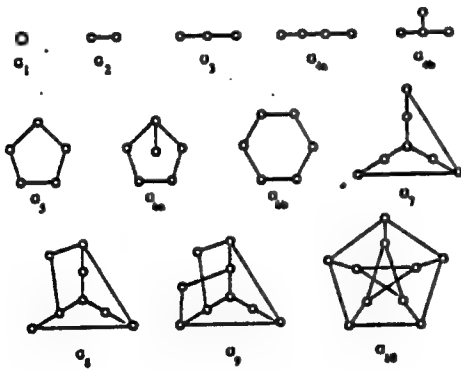
Figure 1: Extremal graphs with $v \leq 10$

Theorem 3.3. The values of $f(v)$ for $11 \leq v \leq 20$ are as follows:

$$f(11) = 16 \quad f(12) = 18 \quad f(13) = 21 \quad f(14) = 23 \quad f(15) = 26$$
$$f(16) = 28 \quad f(17) = 31 \quad f(18) = 34 \quad f(19) = 38 \quad f(20) = 41$$

We next define a restricted type of tree; many of the proofs in the next section rely on the presence of these trees in extremal graphs. Consider a vertex z of maximum degree Δ a (C_3, C_4)-free graph G. Let the neighborhood of z be $N(z) = \{z_1, z_2, \ldots, z_\Delta\}$. Clearly $N(z)$ is an independent set of vertices. Furthermore, the sets of vertices $N(z_i) - \{z\}$, $1 \leq i \leq \Delta$, are pairwise disjoint; otherwise there would be a quadrilateral in G. This motivates the notion of an (m,n)-star $S_{m,n}$, which is defined to be the tree in which the root (center) has m children, and each of the root's children has n children, all of which are leaves. The subtree containing a child of the root and all its n children is called a branch of $S_{m,n}$. We will make use of the following easily established facts: (1) $|V(S_{m,n})| = 1 + m + mn$ and $|E(S_{m,n})| = m + mn$; (2) Every (C_3, C_4)-free graph G with at least 5 vertices, $\Delta(G) = \Delta$, and $\delta(G) = \delta$, contains $S_{\Delta, \delta-1}$; (3) In any (C_3, C_4)-free graph G containing an (m,n)-star S, no vertex in $G - S$ can be adjacent to two siblings S. In fact, every set of siblings in S has a unique common neighbor, namely, their parent.

Proposition 2.6. For all (C_3, C_4)-free graphs G, $v \geq 1 + \Delta\delta \geq 1 + \delta^2$.

Proposition 2.7. For all (C_3, C_4)-free graphs G on $v \geq 1$ vertices and e edges, $\delta \geq e - f(v-1)$ and $\Delta \geq \lceil 2e/v \rceil$.

Proposition 2.8. For all $v \geq 1$, we have $e \geq 1 + \lceil 2f(v)/v \rceil (f(v) - f(v-1))$.

9

Proof. We look at several specific cases to illustrate the techniques involved.
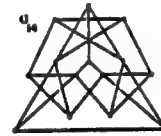
$f(14) = 23$. The graph G_{14} demonstrates that $f(14) \geq 23$. Assume there exists an extremal graph G with 24 edges. Propositions 2.6 and 2.7 imply that $\delta = 3$ and $\Delta = 4$. Thus G has 8 vertices of degree 3 and 6 vertices of degree 4. Further, G cannot contain pair of adjacent vertices x and y each having degree 3, since $G - \{x, y\}$ would have 19 edges, contradicting $f(12) = 18$. We may then conclude that every edge of G joins a degree vertex with a degree 4 vertex. If for each degree 3 vertex we represent its neighborhood as a triple, then the existence of G implies the existence of a design of 8 triples on 6 elements (the six vertices of degree 4), where each element occurs in 4 triples and each distinct pair of elements occurs in at most 1 triple. But since 6 elements constitute 15 distinct pairs and each triple specifies 3 pairs, each 8 triples will specify $8 \times 3 = 24$ pairs, and therefore cannot exist. Thus, $f(14) < 24$.
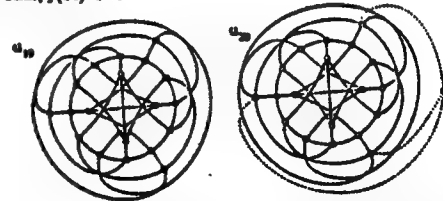
$f(19) = 38$ and $f(20) = 41$. The graph G_{19} shows that $f(19) \geq 38$. Since the outermost 3 vertices (the vertices outside the dodecagon) are mutually distance 3 apart, we can construct G_{20} by adding a vertex to G_{19} that is adjacent to the 3 outer vertices. (In the figure, dashed lines are used to show the edges from the 20th vertex to G_{19}.) Thus $f(20) \geq |E(G)| = 41$. The upper bounds are obtained by applying Proposition 2.8.

It can be noted that G_{10} is the Robertson graph – the unique $(4,5)$-cage. It follows that G_{10} and G_{20} are the unique extremal graphs of orders 10 and 20, respectively.

Theorem 3.4. The values of $f(v)$ for $21 \leq v \leq 24$ are as follows:
$$f(21) = 44 \quad f(22) = 47 \quad f(23) = 50 \quad f(24) = 54$$

Proof. Again, to illustrate the techniques involved, we give the proof that $f(22) = 47$. First of all, a $\{C_3, C_4\}$-free graph of order 22 and size 47 is constructed, showing that $f(22) \geq 47$. Suppose there exists a $\{C_3, C_4\}$-free graph G with $v = 22$, $e = 48$. By Proposition 2.7, $\delta \geq 4$ and $\Delta \geq 5$. Then $\Delta = 5$, for otherwise there is a $(6,3)$ star in G, and such a tree has more than 22 vertices; therefore, G contains 14 vertices of degree 4 and 8 vertices of degree 5. This in turn implies that there are at least 8 edges among the degree 4 vertices. Since any graph with 14 vertices and at least 8 edges must contain a path of length 3, there must be at least one degree 4 vertex adjacent to two other degree 4 vertices; thus there is a P_3 in G each of whose vertices has degree 4. However, if there is a path P on 3 vertices of degree 4, then $G - V(P)$ is the Robertson graph. Each of the pendant vertices in P, x and y, has 3 neighbors in the Robertson graph that are mutually distance 3 apart. But the Robertson graph has only one such set of 3 vertices. Therefore, G cannot contain a P_3 of degree 4 vertices, giving us a contradiction. Therefore, $f(22) \leq 47$.

This paper gives exact values of $f(v)$ for $1 \leq v \leq 24$. It is also noted that $f(50) = 175$, the extremal graph being the Hoffman-Singleton graph. The paper also gives constructive lower bounds for $f(v)$ for $25 \leq v \leq 200$; these are found using an algorithm that combines hill-climbing and backtracking techniques. For instance, at one point a $\{C_3, C_4\}$-free graph of order 96 and size 397 was found; adding an isolated vertex gave a $\{C_3, C_4\}$-free graph of order 97 and size 397. Backtracking applied to this graph yielded a $\{C_3, C_4\}$-free graph of the same order with 403 edges. Hill-climbing from that point resulted in the addition of one more edge, giving a $\{C_3, C_4\}$-free graph G of order 97 and size 404. Thus, $f(97) \geq 404$. Further hill-climbing rearranged the edges of G so that some vertex had degree 6; removing this vertex then gave a $\{C_3, C_4\}$-free graph of order 96 and size 398, thereby improving the lower bound for $f(96)$.

12

"Random Graph Processes with Degree Restrictions"

Andrzej Rucinski, Emory University
Department of Mathematics and Computer Science, Atlanta, GA

Random Graph Processes
with degree restrictions

Andrzej Ruciński
Poznań

Nick C. Wormald
Melbourne

Random graph process

$(E_1, \ldots, E_{\binom{n}{2}})$

$\left(\binom{n}{2}!\right)^{-1}$
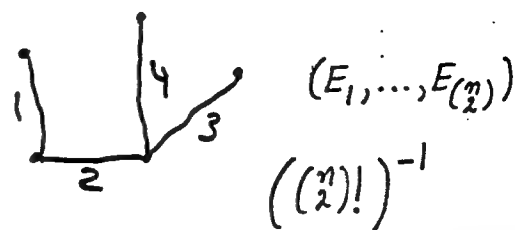
Degree restriction $\Delta(G) \leqslant d$

$d=2$

or

or

$(E_1, \ldots, E_w)$
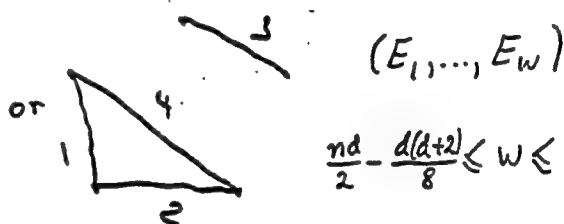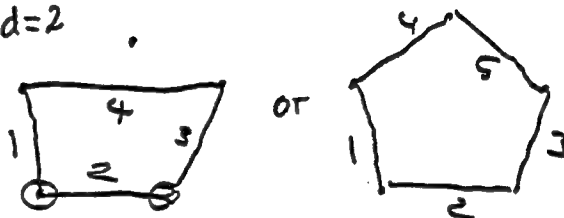
$\frac{nd}{2} - \frac{d(d+2)}{8} \leqslant w \leqslant \left\lfloor \frac{dn}{2} \right\rfloor$

---

3 major differences:
$1°$ many final stages
$2°$ the length varies
$3°$ not an equiprobable space
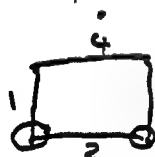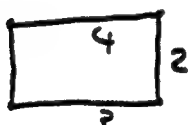
$\frac{1}{10} \frac{1}{9} \frac{1}{6} \frac{1}{3}$ $\frac{1}{10} \frac{1}{9} \frac{1}{8} \frac{1}{3}$

Focus on final stage:
a maximal graph with $\Delta(G)=d$
   d-maximal graph
saturated vts (degree d)
unsaturated vts (degree < d)

Unsaturated vts form a clique,
so there are $\leqslant d$ of them.
Let $U=U(d,n) = \#$ unsaturated vts
                at the end

Erdös asked

$\lim_{n \to \infty} P(U=x), x=0,1,\ldots,d$?

Nontrivial even for $d=2$

$$P(U=0) = \frac{11}{15} \xrightarrow{n=4} \frac{17}{27} \nearrow .879 \xrightarrow{n=5} \approx .9$$
n=500 n=30,000

$P(U=0) \to 1$?

⑤

Theorem (R., Wormald, 1992)

$\forall d$ the process saturates a.s.,

i.e. $P(U=0) \to 1$ if nd even

$P(U=1) \to 1$ if nd odd ∴

It contrasts with the equiprobable space of d-maximal graphs, where

$\forall d \geq 2$, dn even,

$$P(U=i) \to a_i > 0 \quad i = 0, 1, 2,$$
$$a_0 + a_1 + a_2 = 1$$

Structural results only for $d=2$

$$EC \leq \log n + 3$$

$$\# C_l \overset{\sigma}{\leadsto} \text{Poisson}$$

$$E\#C_3 \sim \frac{1}{2} \int_0^\infty \frac{(\log(1+x))^2}{xe^x} dx \approx$$

$$.18873534935777885350 \neq \frac{1}{6}$$

⑦

The idea of proof:

study # isolates I_t

$d=2$

$d \geq 2$ induction on d

$\mapsto$ (d-1)-process
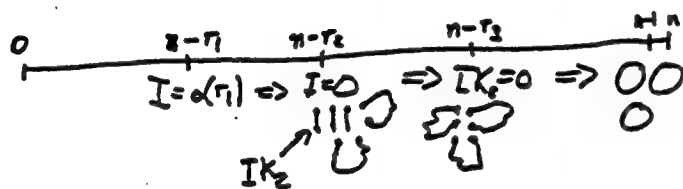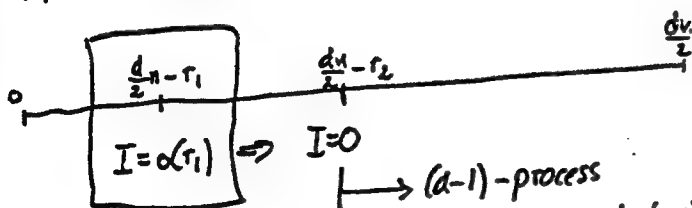
Quite messy : 1° forbidden pairs at start
2° nonuniform degree bounds

Lemma $P(u,v)$ – probability that vertex 1 remains isolated until time v, provided it was isolated at time u. Then

$$P(u,v) = O\left(\frac{n/2 - v}{n/2 - u}\right)$$

Proof $P(u,v) = \prod_{t=u}^{v-1} P(t, t+1)$

H_t – the event that 1 is isolated in (

$$P(t, t+1) = E\left(E\left(Ind(H_{t+1})|G_t\right)|H_t\right)$$

$$= E\left(1 - \frac{U_t - 1}{\binom{U_t}{2} - F_t} \Big| H_t\right) \leq$$

$$\leq \exp\left(-\frac{2}{n - 2t} + O\left(\frac{1}{n^2}\right)\right)$$

Lemma
$$I_{\lfloor dn/2 \rfloor - n^{9/10}} = O\left(\frac{n^{9/10}}{\log n}\right)$$

Outline of proof:

$$E(I_{t+1} - I_t \mid G_t) = ?$$

$$= \frac{-2\binom{I_t}{2} - I_t(U_t - I_t)}{\binom{U_t}{2} - I_t} \sim -\frac{2 I_t}{U_t}$$

Define Doob's martingale

$$X_i = E(I_{t+\Delta t} - I_t \mid G_t, E_{t+1}, \ldots, E_{t+i})$$

$$i = 0, \ldots, \Delta t$$

to show the sharp concentration

$$I_{t+\Delta t} - I_t \sim E(I_{t+\Delta t} - I_t \mid G_t)$$

by Azuma's inequality

But

$$dn - 2t \geqslant d I_t + (U_t - I_t)$$
$$\underset{(d=2)}{=\!=}$$

$$U_t \leq dn - 2t - (d-1) I_t$$
$$\underset{(d=2)}{=\!=}$$

So
$$E(I_{t+\Delta t} - I_t \mid G_t) \lesssim -\frac{2 \Delta t \, I_t}{dn - 2t - (d-1)I_t}$$

Define $b = b(x)$, $0 \leqslant x < \frac{d}{2}$

by

$$(*) \quad b' = \frac{-2b}{d - 2x - (d-1)b}, \quad b(0) = 1$$

$b(x)$ should well approximate
an upper bound on $\frac{I_t}{n}$, $x = \frac{t}{n}$

We justify this by partitioning

and using induction

$$\Delta \sim n^{1/4}$$

The asymptotic solution to (*), as $x \to \frac{d}{2}$, is

$$b(x) \sim \frac{-(d-2x)}{(d-1)\log\left(\frac{d}{2}-x\right)}$$

Finally,

$$I_s \leq nb\left(\frac{s}{n}\right) + \underline{O(n^{8/9})} \sim$$

$$\sim \frac{-n\left(d-2\frac{s}{n}\right)}{(d-1)\log\left(\frac{d}{2}-\frac{s}{n}\right)} = \frac{-2n^{9/10}}{(d-1)\log n^{-1/10}} =$$

$$= \frac{20\, n^{9/10}}{(d-1)\log n} \quad ∎$$

Open problem

How far apart are max and min of

$$P\left(G_{\lfloor\frac{dn}{2}\rfloor} = G\right)$$

a) over all d-maximal G's

b) over all d-regular G's

$$?$$

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Kokou Abalo
Clemson University
Dept. of Math. Sci.
O-313 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Ed Allen
Wake Forest Univ.
Dept. of Math. & Comp. Sci.
Winston-Salem, NC 27107
E-Mail: allene@mthcsc.wfu.edu

Palsun Anand
Clemson University
252 Rockcreek Rd.
Clemson, SC 29631
E-Mail: panand@hubcap.clemson.edu

Alok Aurovillian
Clemson University
813 College Ave.
Clemson, SC 29631
E-Mail: alok@cs.clemson.edu

Frank Bartek
Clemson University
Dept. of Math. Sci.
O-7 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Bruce Berentsen
Clemson University
118 Pleasant View Dr.
Clemson, SC 29631
E-Mail:

Jean R.S. Blair
University of Tennessee
Dept. of Comp. Sci.
107 Ayres Hall
Knoxville, TN 37996-1301
E-Mail: blair@cs.utk.edu

Jay Boland
East TN State Univ.
Dept. of Math.
Johnson City, TN 37614
E-Mail:

Patricia Broadwell
Clemson University
Dept. of Math. Sci.
O-13 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Rod Canfield
University of GA
Dept. of Comp. Sci.
Athens, GA 30602
E-Mail: erc@pollux.cs.uga.edu

Steven Cater
University of GA
Dept. of Comp. Sci.
415 Boyd Grad Studies
Athens, GA 30602
E-Mail: cater@pollux.cs.uga.edu

Mark Cawood
Clemson University
Dept. of Math. Sci.
O-9 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Shuchi Chawla
Clemson University
202 Edgewood Ave., Apt. 11
Clemson, SC 29631
E-Mail: shuchic@cs.clemson.edu

David Cribb
Clemson University
Dept. of Math. Sci.
O-11 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Sandy Davis
Clemson University
Dept. of Math. Sci.
O-13 Martin Hall
Clemson, SC 29634-1907
E-Mail:

James E. Dawsey
Clemson University
14 Green Glenn Apts.
Pendleton, SC 29670
E-Mail:

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Nate Dean
Bellcore
Room 2M-389
445 South St.
Morristown, NJ 07960-1910
E-Mail: nate@bellcore.com

==================================

Donald Dearholt
MS State Univ.
Dept. of Comp. Sci.
P.O. Drawer CS
Mississippi State, MS 39762-5623
E-Mail: dearholt@cs.msstate.edu

==================================

Gopinath Devalcheruvi
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail: gopi@cs.clemson.edu

==================================

Gayla Domke
Georgia State Univ.
Dept. of Math & Comp. Sci.
University Plaza
Atlanta, GA 30303
E-Mail: matgsd@gsuvm1.bitnet

==================================

Jean Dunbar
Converse College
Math Dept.
Spartanburg, SC 29301
E-Mail:

==================================

Ivy Dymacek
Washington & Lee Univ.
Dept. of Mathematics
Lexington, VA 24450
E-Mail:

==================================

Wayne M. Dymacek
Washington & Lee Univ.
Dept. of Mathematics
Lexington, VA 24450
E-Mail: dymacek.w.m@P9955.wlu.edu

==================================

Murali Earagolla
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail: mearago@cs.clemson.edu

==================================

Nancy Eaton
Emory University
Dept. of Math & Comp. Sci.
Atlanta, GA 30322
E-Mail: nancy@mathcs.emory.edu

==================================

Markus Emsermann
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==================================

R.C. Entringer
Univ. of New Mexico
Dept. of Mathematics
Albuquerque, NM 87131
E-Mail: entring@gauss.unm.edu

==================================

Chris Fisher
Univ. of Regina, CANADA
Dept. of Math. Sci.
O-12 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==================================

Sandy Fulmer
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail:

==================================

Banumathi Ganesan
Clemson University
320 Sloan St. #15
Clemson, SC 29631
E-Mail:

==================================

M. Gopichand
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail:

==================================

Daniel Gordon
University of GA
Dept. of Comp. Sci.
Athens, GA 30602
E-Mail:

==================================

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Ron Gould
Emory University
Dept. of Math & Comp. Sci.
Atlanta, GA 30322
E-Mail: rg@mathcs.emory.edu

==

Judith E. Green
Clemson University
P.O. Box 994
Central, SC 29630
E-Mail:

==

Raymond Griffith
Clemson University
Dept. of Math. Sci.
O-313 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==

David Grose
Clemson University
413 Lindsay Rd., Apt. #2
Clemson, SC 29631
E-Mail:

==

J. Guldne
Clemson University
100 Riding Rd.
Clemson, SC 29631
E-Mail: jguldne@eng.clemson.edu

==

William Hare
Clemson University
Dept. of Math. Sci.
O-19 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==

April Haynes
Clemson University
Dept. of Math. Sci.
O-102 Martin Hall
Clemson, SC 29634-1907
E-Mail: ahayne@clemson.edu

==

Teresa W. Haynes
East TN State Univ.
Dept. of Comp. Sci.
Johnson City, TN 37614
E-Mail: i01twh@etsu

==

Sandee Hedetniemi
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail:

==

Steven T. Hedetniemi
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail: hedet@cs.clemson.edu

==

Angela Holland
Clemson University
114 Wigington St.
Clemson, SC 29631
E-Mail: alhollan@cs.clemson.edu

==

Daryl Holoman
Clemson University
Dept. of Comp. Sci.
G-31 Jordan Hall
Clemson, SC 29634
E-Mail: dholoma

==

Fred Howard
Wake Forest Univ.
Dept. of Math. & Comp. Sci.
Winston-Salem, NC 27107
E-Mail:

==

Michael S. Jacobson
University of Louisville
Dept. of Mathematics
Louisville, KY 40292
E-Mail: msjaco01@ulkyvx

==

David John
Wake Forest Univ.
Dept. of Math. & Comp. Sci.
Winston-Salem, NC 27107
E-Mail:

==

Kim Jonas
Univeristy of SC
Dept. of Math.
Columbia, SC 29208
E-Mail:

==

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

James A. Knisely
Clemson University
5B Christopher St.
Greenville, SC 29609
E-Mail:

==

Debra Knisley
East TN State Univ.
Mathematics Dept.
Johnson City, TN 37601
E-Mail: gibi@etsu.bitnet

==

Michael Kostreva
Clemson University
Dept. of Math. Sci.
O-307 Martin Hall
Clemson, SC 29634-1907
E-Mail: flstgla@clemson

==

Y. Amar Kumar
Clemson University
109-1 Earle St.
Clemson, SC 29631
E-Mail: amar@cs.clemson.edu

==

Judy Lalani
Tri County Tech
9-C Daniel Dr.
Clemson, SC 29631
E-Mail:

==

Renu Laskar
Clemson University
Dept. of Math. Sci.
O-16 Martin Hall
Clemson, SC 29634-1907
E-Mail: rclsk@clemson.edu

==

Linda M. Lawson
East TN State Univ.
Dept. of Math.
Johnson City, TN 37614
E-Mail:

==

Marc J. Lipman
Office of Naval Research
Code 1111
Arlington, VA 22217-5000
E-Mail: lipman@ocnr-hq.navy.mil

==

Rainey Little
MS State Univ.
Dept. of Comp. Sci.
P.O. Drawer CS
Mississippi State, MS 39762-5623
E-Mail: little@cs.msstate.edu

==

Dimitmi Logofet
USSR Academy of Sciences
3 Pyzhensky Lane
Moscow, 109107
USSR,
E-Mail:

==

Robin Lougee-Heimer
Clemson University
Dept. of Math. Sci.
O-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==

Lisa Markus
Vanderbilt University
Department of Mathematics
Box 1543 Station B
Nashville, TN 37235
E-Mail:
lmarkus@athena.cas.vanderbilt.edu

==

Brad Martin
Washington & Lee Univ.
Dept. of Mathematics
Lexington, VA 24450
E-Mail: martin.b.d@P9955.wlu.edu

==

Travis Mauldin
Clemson University
616 Railroad St.
Pickens, SC 29671-2803
E-Mail: rmauldi@cs.clemson.edu

==

Robert McDaniel
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail:

==

Buck McMorris
University of Louisville
Dept. of Mathematics
Louisville, KY 40292
E-Mail: frmcmo01@ulkyvx.bitnet

==

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Alice McRae
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail: amcrae@clemson.edu

Seiuivas Nithyanandam
Clemson University
320 Sloan St. #18
Clemson, SC 29631
E-Mail: snithya@engr.clemson.edu

M. Medikonda
Clemson University
411-4 Lindsay Rd.
Clemson, SC 29631
E-Mail: murali@cs.clemson.edu

Tim Niznik
Clemson University
Dept. of Math. Sci.
E-304 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Deborah Meilhammer
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Susan Patterson
Clemson University
Dept. of Math. Sci.
E-304 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Ted Monroe
Wofford College
108 Woodcreek Dr.
Spartanburg, SC 29303
E-Mail:

Barry Piazza
Southern MS
Dept. of Math.
Southern Station Box 5045
Hattiesburg, MS 39406-5045
E-Mail: piazza@usmcp6.bitnet

Donna Mouritzen
Clemson University
Dept. of Math. Sci.
O-10 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Dan Pillone
Clemson University
Knoll Apt. 7B, 123 Clg Heights Blvd
Clemson, SC 29631
E-Mail:

Nishad V. Mulye
Clemson University
115-1 Hill Crest Ave.
Clemson, SC 29631
E-Mail:

Carl Pomerance
University of GA
Dept. of Math
Athens, GA 30602
E-Mail:

Matthew Myers
Clemson University
Dept. of Math. Sci.
E-304 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Dorh. Potluri
Clemson University
113-3 Earle St.
Clemson, SC 29631
E-Mail: dpotlur@cs.clemson.edu

Jennifer Newton
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Gil Proctor
Clemson University
Dept. of Math. Sci.
O-102 Martin Hall
Clemson, SC 29634-1907
E-Mail: proctor@clemson.edu

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Douglas F. Rall
Furman University
Dept. of Math.
Greenville, SC 29613
E-Mail: rall@frmnvax1.bitnet

Andrzej Rucinski
Emory University
Dept. of Math & Comp. Sci.
Atlanta, GA 30322
E-Mail: andrzej@mathcs.emory.edu

===================================

Bobby Rampey
Furman University
27 Meyers Ct.
Greenville, SC 29609
E-Mail:

Scott Ruzycki
Clemson University
Comp. Info. Sys.
Box 4677
Clemson, SC 29632
E-Mail: sruzyck

===================================

Amish Ray
Clemson University
211-2 Kelly Rd.
Clemson, SC 29631
E-Mail: amish@cs.clemson.edu

Matthew Saltzman
Clemson University
Dept. of Math. Sci.
O-223 Martin Hall
Clemson, SC 29634-1907
E-Mail:

===================================

Ron Read
University of Waterloo
Dept. of Comb. & Optim.
Waterloo, Ontario
CANADA, N2L 3G1
E-Mail: rread@watdcs.uwaterloo.ca

Moitri Sarker
Clemson University
Dept. of Math. Sci.
M-303 Martin Hall
Clemson, SC 29634-1907
E-Mail:

===================================

Virginia Rice
Kennesaw State College
Dept. of Mathematics
Marietta, GA 30061
E-Mail:

Dinesh Sarvate
College of Charleston
Math Dept.
Charleston, SC 29424
E-Mail:

===================================

Richard D. Ringeisen
Clemson University
Dept. of Math. Sci.
O-103 Martin Hall
Clemson, SC 29634-1907
E-Mail: rdrng@clemson.edu

Edward R. Scheinerman
Johns Hopkins Univ.
Mathematical Sciences Dept.
Baltimore, MD 21218
E-Mail: ers@cs.jhu.eud

===================================

Fred Roberts
DIMACS
Rutgers Univ.
P.O. Box 1179
Piscataway, NJ 08855-1179
E-Mail: froberts@dimacs.rutgers.edu

Mary Beth Searcy
Clemson University
Dept. of Math. Sci.
M-303 Martin Hall
Clemson, SC 29634-1907
E-Mail:

===================================

Robert W. Robinson
University of GA
Compter Sci. Dept.
415 GSRC
Athens, GA 30602
E-Mail: rwr@pollux.cs.uga.edu

Franklin Shobe
Clemson University
Dept. of Math. Sci.
O-2 Martin Hall
Clemson, SC 29634-1907
E-Mail: fshobe@clemson.bitnet

===================================

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Robert Simms
Clemson University
Dept. of Math. Sci.
E-304 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Kanupriya Tewary
Clemson University
Dept. of Comp. Sci.
G-16 Jordan Hall
Clemson, SC 29634
E-Mail: kanu@cs

M. Sinivadavao
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
Clemson, SC 29634
E-Mail: srini

Cary Timar
Vanderbilt University
Mathematics Dept.
1326 Stevenson Center
Nashville, TN 37235
E-Mail:

Jim Soltys
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Darol Timberlake
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Michael Springfield
Clemson University
1016 Highland Rd.
Easley, SC 29640
E-Mail: mspring

Charles Wallis
Clemson University
Dept. of Math. Sci.
M-304 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Lionel Stewart
Clemson University
Dept. of Comp. Sci.
G-31 Jordan Hall
Clemson, SC 29634
E-Mail: lionels

Ruth Wassermann
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Joseph Straight
SUNY at Fredonia
Dept. of Math. & Comp. Sci.
Fredonia, NY
E-Mail: straight@fredonia

Margaret Wiecek
Clemson University
Dept. of Math. Sci.
O-208 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Charles Suffel
Stevens Institute
730 Hudson St., Apt. 53
Hoboken, NJ 07030
E-Mail:

Ben Williams
Clemson University
Dept. of Math. Sci.
O-313 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Frances Sullivan
Clemson University
Dept. of Math. Sci.
O-18 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Jeff Williams
Clemson University
Dept. of Math. Sci.
E-8 Martin Hall
Clemson, SC 29634-1907
E-Mail:

Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
Attendees

Jody Wilson
Clemson University
294 Brookwood Dr., Apt. J
Seneca, SC 29678-2451
E-Mail:

==

Loretta Winston
Clemson University
Dept. of Comp. Sci.
College of Nursing Bldg.
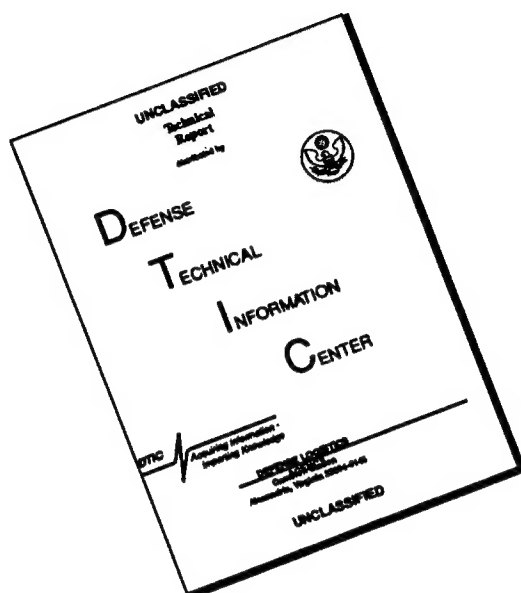Clemson, SC 29634
E-Mail: lwinston@cs.clemson.edu

==

Ewa Wojcicka
College of Charleston
Math Dept.
Charleston, SC 29424
E-Mail:

==

Deda Zheng
Clemson University
Dept. of Math. Sci.
O-7 Martin Hall
Clemson, SC 29634-1907
E-Mail:

==

DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.